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CYCLICALLY REDUCED ELEMENTS IN COXETER GROUPS

BY TIMOTHÉE MARQUIS

ABSTRACT. – Let W be a Coxeter group. We provide a precise description of the conjugacy classes in W, in the spirit of Matsumoto's theorem. This extends to all Coxeter groups an important result on finite Coxeter groups by M. Geck and G. Pfeiffer from 1993. In particular, we describe the cyclically reduced elements of W, thereby proving a conjecture of A. Cohen from 1994.

RÉSUMÉ. – Soit W un groupe de Coxeter. Nous donnons dans cet article une description précise des classes de conjugaison dans W, dans l'esprit du théorème de Matsumoto. Ceci étend à tous les groupes de Coxeter un résultat important sur les groupes de Coxeter finis de M. Geck et G. Pfeiffer datant de 1993. En particulier, nous décrivons les éléments cycliquement réduits de W, établissant ainsi une conjecture d'A. Cohen datant de 1994.

1. Introduction

Let (W, S) be a Coxeter system. By a classical result of J. Tits ([26]), also known as Matsumoto's theorem (see [20]), any given reduced expression of an element $w \in W$ can be obtained from any other expression of w by performing a finite sequence of braid relations and ss-cancelations (i.e., replacing a subword (s, s) for some $s \in S$ by the empty word). In particular, this yields a very simple and elegant solution to the word problem in Coxeter groups.

The conjugacy problem for Coxeter groups was solved about 30 years later, first thanks to the work of G. Moussong ([21]) that yields an exponential time algorithm, and then by D. Krammer in his thesis from 1994 (published in [16]): there exists a cubic algorithm deciding whether two words on the alphabet S determine conjugate elements of W. However, Moussong's and Krammer's solutions do not provide a sequence of "elementary operations" to pass from one word to the other, as do the braid relations and ss-cancelations in Matsumoto's theorem.

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In this paper, we address the following long-standing open question on Coxeter groups: *Is there an analogue of Matsumoto's theorem for the conjugacy problem in Coxeter groups?*

A very natural elementary operation on words to consider for the conjugacy problem is that of cyclic shift: by extension, we say that an element $w' \in W$ is a *cyclic shift* of some $w \in W$ if there is some reduced decomposition $w = s_1 \dots s_d$ ($s_i \in S$) of w such that either $w' = s_2 \dots s_d s_1$ or $w' = s_d s_1 \dots s_{d-1}$. Such operations are, however, not sufficient to describe conjugacy classes in general, as for instance illustrated by the Coxeter group $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$ of type A_2, in which the simple reflections s and t are conjugate, but cannot be obtained from one another through a sequence of cyclic shifts. Nonetheless, in the terminology of [11, Chapter 3], the elements $w := s$ and $w' := t$ are *elementarily strongly conjugate*, meaning that $\ell_S(w) = \ell_S(w')$ and that there exists some $x \in W$ with $w' = x^{-1}wx$ such that either $\ell_S(x^{-1}w) = \ell_S(x) + \ell_S(w)$ or $\ell_S(wx) = \ell_S(w) + \ell_S(x)$.

Motivated by the representation theory of Hecke algebras, M. Geck and G. Pfeiffer proved in [10] that if W is finite, then for any conjugacy class $\mathcal{O}$ in W,

1. any $w \in \mathcal{O}$ can be transformed by cyclic shifts into an element w' of minimal length in $\mathcal{O}$, and
2. any two elements w, w' of minimal length in $\mathcal{O}$ are *strongly conjugate*, i.e., there exists a sequence $w = w_0, \dots, w_n = w'$ of elements of W such that w_{i-1} is elementarily strongly conjugate to w_i for each $i = 1, \dots, n$.

Together with S. Kim, they later generalized this theorem (see [9]) to the case of δ-twisted conjugacy classes for some automorphism δ of (W, S), that is, when $\mathcal{O}$ is replaced by $\mathcal{O}_\delta = \{\delta(v)^{-1}wv \mid v \in W\}$ for some $w \in W$. The proofs in [10] and [9] involve a case-by-case analysis, with the help of a computer for the exceptional types. In [14], X. He and S. Nie gave a uniform (and computer-free) geometric proof of that theorem, which they later generalized, in [15], to the case of an affine Coxeter group W. In addition, they showed (for W affine) that

(3) if $\mathcal{O}$ is straight, then any two elements w, w' of minimal length in $\mathcal{O}$ are conjugate by a sequence of cyclic shifts,

where $\mathcal{O}$ is *straight* if it contains a straight element $w \in W$, that is, such that $\ell_S(w^n) = n\ell_S(w)$ for all $n \in \mathbb{N}$ (equivalently, every minimal length element of $\mathcal{O}$ is straight, see Lemma 2.4). Note that the straight elements in an arbitrary Coxeter group were characterized in [19, Theorem D]; these elements play an important role in the study of affine Deligne-Lusztig varieties (see [13]), and also exhibit very useful dynamical properties (see e.g., [18] or [5]). Similar statements to (1) and (2) above were further obtained for an arbitrary Coxeter group W, but when $\mathcal{O}$ is replaced by some "partial" conjugacy class $\mathcal{O} = \{v^{-1}wv \mid v \in W_I\}$, for some finite standard parabolic subgroup $W_I \subseteq W$ (see [12] and [23]). Finally, we showed in [19, Theorem A] that for a certain class of Coxeter groups that includes the right-angled ones, (1) and (2) hold using only cyclic shifts.

In this paper, we prove the statements (1), (2) and (3) in full generality, namely, for an arbitrary Coxeter group W. Moreover, we actually prove a much more precise version of (2) by introducing a refined notion of "strong conjugation," which we call "tight conjugation" (see

Definition 3.4)—in particular, if two elements are tightly conjugate, then they are strongly conjugate; when W is finite, the two notions coincide. Here is our main theorem.

THEOREM A. – *Let (W, S) be a Coxeter system. Let $\mathcal{O}$ be a conjugacy class of W, and let $\mathcal{O}_{\min}$ be the set of minimal length elements of $\mathcal{O}$. Then the following assertions hold:*

1. *For any $w \in \mathcal{O}$, there exists an element $w' \in \mathcal{O}_{\min}$ that can be obtained from w by a sequence of cyclic shifts.*
2. *If $w, w' \in \mathcal{O}_{\min}$, then w and w' are tightly conjugate.*
3. *If $\mathcal{O}$ is straight, then any two elements $w, w' \in \mathcal{O}_{\min}$ are conjugate by a sequence of cyclic shifts.*

Note that the proof of Theorem A uses the results of [9] (or [14]), but does not rely on [15]. In particular, we give an alternative, shorter proof that affine Coxeter groups satisfy Theorem A.

Recall that an element $w \in W$ is *cyclically reduced* if $\ell_S(w') = \ell_S(w)$ for every $w' \in W$ obtained from w by a sequence of cyclic shifts. Often, this terminology is used instead for elements of minimal length in their conjugacy class. An important reformulation of Theorem A(1) is that these two notions in fact coincide.

COROLLARY B. – *An element $w \in W$ is cyclically reduced if and only if it is of minimal length in its conjugacy class.*

This proves a conjecture of A. Cohen (see [6, Conjecture 2.18]).

The proof of Theorem A is of geometric nature, and uses the Davis complex X of (W, S)—here, we assume that S is finite, a safe assumption for the study of Theorem A (see Remark 6.1). This is a CAT(0) cellular complex on which W acts by cellular isometries. For instance, if W is affine, then X is just the standard geometric realization of the Coxeter complex Σ of (W, S), and the CAT(0) metric $\mathrm{d}\colon X \times X \to \mathbb{R}_+$ is the usual Euclidean metric (see Example 2.7). For an element $w \in W$, the subset $\mathrm{Min}(w) \subseteq X$ of all $x \in X$ such that $\mathrm{d}(x, wx)$ is minimal will play an important role; it will also be crucial to investigate its combinatorial analogue $\mathrm{CombiMin}(w)$ (see §4), as highlighted in Remark 5.7. As a byproduct of our proofs, we are able to relate these two notions of "minimal displacement set" for w (see §7).

COROLLARY C. – *Let $w \in W$. Then* $\mathrm{Min}(w) \subseteq \mathrm{CombiMin}(w)$*, and* $\mathrm{CombiMin}(w)$ *is at bounded Hausdorff distance from* $\mathrm{Min}(w)$.

Note that, while $\mathrm{Min}(w)$ is always connected (in the CAT(0) sense), its combinatorial analogue $\mathrm{CombiMin}(w)$ need not be (gallery-)connected, and this is precisely the reason why cyclic shifts are not sufficient to describe the conjugacy classes in W, and why one needs to also consider "tight conjugations" (see Remark 4.6).

We conclude the introduction with a short roadmap to the proof of Theorem A. Let $w \in W$, and let $\mathcal{O}$ (resp. $\mathcal{O}_{\min}$) denote its conjugacy class (resp. the set of elements of minimal length in $\mathcal{O}$). Let $\mathrm{Ch}(\Sigma)$ be the set of chambers of the Coxeter complex Σ ($\mathrm{Ch}(\Sigma)$ which can be W-equivariantly identified with the set of vertices $\{v \mid v \in W\}$ of the Cayley graph of (W, S)).