

quatrième série - tome 54 fascicule 3 mai-juin 2021

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

YVES DE CORNULIER

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE
de 1883 à 1888 par H. DEBRAY
de 1889 à 1900 par C. HERMITE
de 1901 à 1917 par G. DARBOUX
de 1918 à 1941 par É. PICARD
de 1942 à 1967 par P. MONTEL

Comité de rédaction au 1^{er} juin 2021

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Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 437 euros.
Abonnement avec supplément papier :
Europe : 600 €. Hors Europe : 686 € (\$ 985). Vente au numéro : 77 €.

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ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 n^{os} / an

SMALL-TIME FLUCTUATIONS FOR THE BRIDGE OF A SUB-RIEMANNIAN DIFFUSION

BY ISMAËL BAILLEUL, LAURENT MESNAGER
AND JAMES NORRIS

ABSTRACT. – We consider small-time asymptotics for diffusion processes conditioned by their initial and final positions, under the assumption that the diffusivity has a sub-Riemannian structure, not necessarily of constant rank. We show that, if the endpoints are joined by a unique path of minimal energy, and lie outside the sub-Riemannian cut locus, then the fluctuations of the conditioned diffusion from the minimal energy path, suitably rescaled, converge to a Gaussian limit. The Gaussian limit is characterized in terms of the bicharacteristic flow, and also in terms of a second variation of the energy functional at the minimal path, the formulation of which is new in this context.

RÉSUMÉ. – Ce travail traite de l’asymptotique en petit temps de processus de diffusions conditionnés par leurs positions initiale et finale, sous l’hypothèse que la diffusivité a une structure sous-riemannienne, possiblement de rang non constant. On démontre que les fluctuations de la diffusion conditionnée, convenablement ré-échelonnée, convergent vers une limite gaussienne, lorsque les positions initiale et finale sont reliées par un unique chemin d’énergie minimale et sont situées hors du cut-locus sous-riemannien. La limite gaussienne est caractérisée en terme du flot bicaractéristique, ainsi qu’en terme de la variation seconde de l’énergie au voisinage d’un chemin minimal, sous une forme nouvelle dans ce contexte.

1. Introduction

Consider a second order differential operator on \mathbb{R}^d in Hörmander’s form ⁽¹⁾

$$(1) \quad \mathcal{L} = \frac{1}{2} \sum_{\ell=1}^m X_{\ell}^2 + X_0,$$

where X_0, X_1, \dots, X_m are vector fields on \mathbb{R}^d . Let us assume for now that

$$(2) \quad X_0, X_1, \dots, X_m \text{ are bounded with bounded derivatives of all orders}$$

Research supported by EPSRC grant EP/103372X/1.

⁽¹⁾ We identify here X_{ℓ} with the differential operator $\sum_{i=1}^d X_{\ell}^i(x) \partial / \partial x^i$.

and that \mathcal{L} satisfies the strong Hörmander condition on \mathbb{R}^d , that is to say,

$$(3) \quad \text{span}\{Y(x) : Y \in \mathcal{A}(X_1, \dots, X_m)\} = T_x \mathbb{R}^d, \quad \text{for all } x \in \mathbb{R}^d.$$

Here $\mathcal{A}(X_1, \dots, X_m)$ denotes the smallest set of vector fields on \mathbb{R}^d containing X_1, \dots, X_m and closed under the commutator product, given by

$$[X, Y](x) = \sum_{i=1}^d X^i(x) \frac{\partial Y}{\partial x^i}(x) - Y^i(x) \frac{\partial X}{\partial x^i}(x).$$

Our main result concerns the small-time fluctuations of diffusion bridges associated to \mathcal{L} . For $x, y \in \mathbb{R}^d$, write $\Omega^{x,y}$ for the set of continuous paths $\omega : [0, 1] \rightarrow \mathbb{R}^d$ such that $\omega_0 = x$ and $\omega_1 = y$. For $\varepsilon > 0$, denote by $\mu_\varepsilon^{x,y}$ the law on $\Omega^{x,y}$ of the diffusion bridge associated to $\varepsilon \mathcal{L}$ starting from x at time 0 and ending at y at time 1.

Given an absolutely continuous path $\omega : [0, 1] \rightarrow \mathbb{R}^d$, it may be that there exists an absolutely continuous path $h : [0, 1] \rightarrow \mathbb{R}^m$ such that, for almost all t ,

$$\dot{\omega}_t = \sum_{\ell=1}^m X_\ell(\omega_t) \dot{h}_t^\ell.$$

Then the energy $I(\omega)$ may be defined by

$$(4) \quad I(\omega) = \inf \int_0^1 |\dot{h}_t|^2 dt,$$

where the infimum is taken over all such paths h . If ω is not absolutely continuous, or there is no such path h , then we set $I(\omega) = \infty$.

In the case where x and y are joined by a unique path γ of minimal energy, we will write $T_\gamma \Omega^{x,y}$ for the set of continuous paths $v : [0, 1] \rightarrow T\mathbb{R}^d$ such that $v_t \in T_{\gamma_t} \mathbb{R}^d$ for all t and $v_0 = v_1 = 0$. Given $\omega \in \Omega^{x,y}$ and $\varepsilon > 0$, define $\sigma_\varepsilon(\omega) \in T_\gamma \Omega^{x,y}$ by

$$\sigma_\varepsilon(\omega)_t = \frac{\omega_t - \gamma_t}{\sqrt{\varepsilon}}.$$

Then define a probability measure $\tilde{\mu}_\varepsilon^{x,y}$ on $T_\gamma \Omega^{x,y}$ by

$$\tilde{\mu}_\varepsilon^{x,y} = \mu_\varepsilon^{x,y} \circ \sigma_\varepsilon^{-1}.$$

The sub-Riemannian cut locus was defined by Bismut [9] in terms of the bicharacteristic flow associated to the principal symbol a of the operator $2\mathcal{L}$, which is given by

$$(5) \quad a(x) = \sum_{\ell=1}^m X_\ell(x) \otimes X_\ell(x).$$

This is reviewed in detail in Section 2. We can now state a version of our main result.

THEOREM 1.1. – *Let \mathcal{L} be a second order differential operator on \mathbb{R}^d of the form (1). Assume that \mathcal{L} satisfies conditions (2) and (3). Let $x, y \in \mathbb{R}^d$. Suppose that there is a unique path γ of minimal energy in $\Omega^{x,y}$ and that (x, y) lies outside the cut locus. Then $\tilde{\mu}_\varepsilon^{x,y}$ converges weakly to a Gaussian probability measure μ_γ on $T_\gamma \Omega^{x,y}$ as $\varepsilon \rightarrow 0$.*

The covariance of the limit measure μ_γ is given in terms of the bicharacteristic flow in Section 2. Theorem 1.1 is proved in Section 3.

This theorem raises some further questions. First, once the global condition is made that γ has minimal energy, it is natural to hope that a suitable modification of the theorem holds under more local hypotheses. In particular, we would seek to drop the strong conditions that the underlying space is \mathbb{R}^d and that the operator coefficients are bounded with bounded derivatives of all orders. Second, in the Riemannian case, the limit Gaussian measure μ_γ can be characterized in terms of the second variation of the energy function at the minimal path. This leads us to seek an analogous intrinsic object in the sub-Riemannian case. In order to address these questions, we now reset to a more general framework.

Let M be a connected C^∞ manifold of dimension d and let a be a C^∞ non-negative quadratic form on the cotangent space T^*M . We assume that a has a sub-Riemannian structure, that is to say, there exist $m \in \mathbb{N}$ and C^∞ vector fields X_1, \dots, X_m on M such that

$$(6) \quad a(\xi, \xi) = \langle \xi, a(x)\xi \rangle = \sum_{\ell=1}^m \langle \xi, X_\ell(x) \rangle^2, \quad \xi \in T_x^*M$$

and such that

$$(7) \quad \text{span}\{Y(x) : Y \in \mathcal{A}(X_1, \dots, X_m)\} = T_x M, \quad \text{for all } x \in M.$$

There is associated to the quadratic form a an energy function I on the set of continuous paths $\Omega = C([0, 1], M)$. While this can be defined as in (4), the following equivalent definition makes clear that I is intrinsic to the quadratic form a . An absolutely continuous path $\omega \in \Omega$ may have a driving path ξ , by which we mean a measurable path ξ in T^*M such that $\xi_t \in T_{\omega_t}^*M$ and $\dot{\omega}_t = a(\omega_t)\xi_t$ for almost all t . Then ω has energy

$$I(\omega) = \int_0^1 \langle \xi_t, a(\omega_t)\xi_t \rangle dt.$$

If ω is not absolutely continuous or has no driving path, then $I(\omega) = \infty$. Write H^x for the subset of Ω consisting of paths of finite energy starting from x . For $x, y \in M$, set

$$H^{x,y} = \{\omega \in H^x : \omega_1 = y\}.$$

It is well known, under the bracket condition (7), that $H^{x,y}$ is non-empty and that the sub-Riemannian distance

$$(8) \quad d(x, y) = \inf_{\omega \in H^{x,y}} \sqrt{I(\omega)}$$

defines a metric compatible with the topology of M .

Our main result concerns the case where x and y are chosen so that I achieves a minimum on $H^{x,y}$ uniquely, say at γ . We will then construct, under a regularity condition on γ , a vector space $T_\gamma H^{x,y}$ of absolutely continuous paths v in TM , with $v_t \in T_{\gamma_t} M$ for all t and $v_0 = v_1 = 0$, along with an equivalence class of norms on $T_\gamma H^{x,y}$, each making $T_\gamma H^{x,y}$ into a Hilbert space. The paths in $T_\gamma H^{x,y}$ can be thought of as the infinitesimal variations of γ in $H^{x,y}$. We will further construct a continuous non-negative quadratic form Q on $T_\gamma H^{x,y}$ such that $Q(v)$ is the minimal second variation of I in the direction v , in a sense to be made precise. These constructions are the content of Section 5.