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RANDOM WALK ON UNIPOTENT MATRIX GROUPS

BY PERSI DIACONIS AND ROBERT HOUGH

ABSTRACT. – We introduce a new method for proving central limit theorems for random walk on nilpotent groups. The method is illustrated in a local central limit theorem on the Heisenberg group, weakening the necessary conditions on the driving measure. As a second illustration, the method is used to study walks on the $n \times n$ uni-upper triangular group with entries taken modulo p. The method allows sharp answers to the behavior of individual coordinates: coordinates immediately above the diagonal require order p^2 steps for randomness, coordinates on the second diagonal require order p steps; coordinates on the kth diagonal require order $p^{\frac{2}{k}}$ steps.

RÉSUMÉ. – Nous introduisons une nouvelle méthode pour prouver les théorèmes limites centraux pour la marche aléatoire sur groupes nilpotents. La méthode est illustrée dans un théorème de la limite centrale locale sur le groupe Heisenberg, affaiblissant les conditions nécessaires sur la mesure sous-jacente. Comme deuxième illustration, la méthode est utilisée pour étudier les marches aléatoires sur le groupe triangulaire des matrices uni-supérieures $n \times n$ avec des entrées prises modulo p. La méthode permet des réponses précises sur le comportement des coordonnées individuelles: les coordonnées immédiatement au-dessus de la diagonale nécessitent un ordre p^2 pour devenir aléatoire, les coordonnées sur la deuxième diagonale nécessitent un ordre de p pas pour converger; les coordonnées sur la k-ième diagonale nécessitent un ordre de magnitude de $p^{\frac{2}{k}}$ pas.

1. Introduction

Let $\mathbb{H}(\mathbb{R})$ denote the real Heisenberg group

$$\mathbb{H}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \tag{1}$$

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Abbreviate $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with $[x, y, z]$, identified with a vector in $\mathbb{R}^3$. Consider a simple random walk on $G = \mathbb{H}(\mathbb{R})$ driven by Borel probability measure μ. For $N \geq 1$, the law of this walk is the convolution power μ^{*N} where, for Borel measures μ, ν on G, and for $f \in C_c(G)$,

$$\langle f, \mu * \nu \rangle = \int_{g,h \in G} f(gh)\, d\mu(g)\, d\nu(h). \tag{2}$$

Say that measure μ is non-lattice (aperiodic) if its support is not contained in a proper closed subgroup of G. For general non-lattice μ of compact support Breuillard [6] uses the representation theory of G to prove a local limit theorem for the law of μ^{*N}, asymptotically evaluating its density in translates of bounded Borel sets. However, in evaluating μ^{*N} on Borel sets translated on both the left and the right he makes a decay assumption on the Fourier transform of the abelianization of the measure μ, and raises the question of whether this is needed. We show that this condition is unnecessary. In doing so we give an alternative approach to the local limit theorem on G treating it as an extension of the classical local limit theorem on $\mathbb{R}^n$. We also obtain the best possible rate. The method of argument is analogous to (though simpler than) the analysis of quantitative equidistribution of polynomial orbits on G from [14].

Recall that the abelianization $G_{\text{ab}} = G/[G, G]$ of G is isomorphic to $\mathbb{R}^2$ with projection $p : G \to G_{\text{ab}}$ given by $p([x, y, z]) = [x, y]$. Assume that the probability measure μ satisfies the following conditions.

i. *Compact support.*

ii. *Centered.* The projection p satisfies

$$\int_G p(g) d\mu(g) = 0. \tag{3}$$

iii. *Full dimension.* Let $\Gamma = \overline{\langle \operatorname{supp} \mu \rangle}$ be the closure of the subgroup of G generated by the support of μ. The quotient G/Γ is compact.

Section 2 gives a characterization of closed subgroups Γ of G of full dimension.

Under the above conditions, the central limit theorem for μ is known. Let $(d_t)_{t>0}$ denote the semigroup of dilations given by

$$d_t([x, y, z]) = [tx, ty, t^2 z] \tag{4}$$

and denote the Gaussian semigroup $(\nu_t)_{t>0}$ defined by its generator (see [6], [26])

$$\begin{aligned} \mathscr{A} f &= \frac{d}{dt}\bigg|_{t=0} \int_{g \in G} f(g) d\nu_t(g) \\ &= \overline{z}\partial_z f(\text{id}) + \overline{xy}\partial_{xy}^2 f(\text{id}) + \frac{1}{2}\overline{x^2}\partial_x^2 f(\text{id}) + \frac{1}{2}\overline{y^2}\partial_y^2 f(\text{id}) \end{aligned} \tag{5}$$

where $\sigma_x^2 = \overline{x^2} = \int_{g=[x,y,z] \in G} x^2 d\mu(g)$ and similarly $\sigma_y^2 = \overline{y^2}$, $\sigma_{xy}^2 = \overline{xy}$, $\overline{z}$. With $\nu = \nu_1$, the central limit theorem for μ states that for $f \in C_c(G)$,

$$\left\langle f, d_{\frac{1}{\sqrt{N}}} \mu^{*N} \right\rangle \to \langle f, \nu \rangle. \tag{6}$$

For $g \in G$ define the left and right translation operators $L_g, R_g : L^2(G) \to L^2(G)$,

$$L_g f(h) = f(gh), \qquad R_g f(h) = f(hg). \tag{7}$$

Our local limit theorem in the non-lattice case is as follows.

THEOREM 1. – *Let μ be a Borel probability measure of compact support on $G = \mathbb{H}(\mathbb{R})$, which is centered and full dimension. Assume that the projection to the abelianization μ_{ab} is non-lattice. Let ν be the limiting Gaussian measure of $d_{\frac{1}{\sqrt{N}}}\mu^{*N}$. For $f \in C_c(G)$, uniformly for $g, h \in G$, as $N \to \infty$,*

$$\left\langle L_g R_h f, \mu^{*N}\right\rangle = \left\langle L_g R_h f, d_{\sqrt{N}}\nu\right\rangle + o_{\mu,f}\left(N^{-2}\right). \tag{8}$$

If the Cramér condition holds:

$$\sup_{\lambda\in\widehat{\mathbb{R}^2},\ |\lambda|>1}\left|\int_{g=[x,y,z]\in G} e^{-i\lambda\cdot(x,y)}d\mu(g)\right| < 1, \tag{9}$$

then uniformly for $g, h \in G$ and Lipschitz $f \in C_c(G)$, as $N \to \infty$

$$\left\langle L_g R_h f, \mu^{*N}\right\rangle = \left\langle L_g R_h f, d_{\sqrt{N}}\nu\right\rangle + O_{\mu,f}\left(N^{-\frac{5}{2}}\right). \tag{10}$$

REMARK. – The rate is best possible as may be seen by projecting to the abelianization. A variety of other statements of the local theorem are also derived, see eqn. (74) in Section 3.

REMARK. – For non-lattice μ, [6] obtains (8) with $h = \mathrm{id}$ and for general h subject to Cramér's condition. A condition somewhat weaker than Cramér's would suffice to obtain (10).

REMARK. – In the case that μ is supported on a closed discrete subgroup or has a density with respect to Haar measure, [1, 2] obtains an error of $O\left(N^{-\frac{5}{2}}\right)$ in approximating $\mu^{*N}(g)$, $g \in \Gamma$.

Our proof of Theorem 1 applies also in the case when μ_{ab} has a lattice component, and gives a treatment which is more explicit than the argument in [1]. To illustrate this, we determine the leading constant in the probability of return to 0 in simple random walk on $\mathbb{H}(\mathbb{Z})$, giving an alternative proof of a result of [19]. Our proof applies equally well to determine the return probability to 0 of lattice random walks with arbitrary finitely supported driving measures.

THEOREM 2. – *Let μ_0 be the measure on $\mathbb{H}(\mathbb{Z})$ which assigns equal probability $\frac{1}{5}$ to each element of the generating set*

$$\left\{\mathrm{id}, \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix}\right\}. \tag{11}$$

As $N \to \infty$, $\mu_0^{*N}(\mathrm{id}) = \frac{25}{16N^2} + O\left(N^{-\frac{5}{2}}\right)$.