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SPLITTING FAMILIES IN GALOIS COHOMOLOGY

BY Cyril DEMARCHE AND Mathieu FLORENCE

ABSTRACT. – Let k be a field, with absolute Galois group Γ. Let A/k be a finite étale group scheme of multiplicative type, i.e., a finite discrete Γ-module. Let $n \geq 2$ be an integer, and let $x \in H^n(k, A)$ be a cohomology class. We show that there exists a countable set I, and a family $(X_i)_{i \in I}$ of (smooth, geometrically integral) k-varieties, such that the following holds: for any field extension ℓ/k, the restriction of x vanishes in $H^n(\ell, A)$ if and only if (at least) one of the X_i has an ℓ-point. In addition, we show that the X_i can be made into an ind-variety. In the case $n = 2$, we note that one variety is enough.

RÉSUMÉ. – Soit k un corps, de groupe de Galois absolu Γ. Soit A/k un schéma en groupes fini étale de type multiplicatif, i.e., un Γ-module fini discret. Soit $n \geq 2$ un entier, et $x \in H^n(k, A)$ une classe de cohomologie. On montre qu'il existe un ensemble dénombrable I, et une famille $(X_i)_{i \in I}$ de k-variétés (lisses, géométriquement intègres) telles que : pour toute extension de corps ℓ/k, la restriction de x s'annule dans $H^n(\ell, A)$ si et seulement si (au moins) une des X_i a un ℓ-point. De plus, on montre qu'on peut choisir les X_i pour qu'elles forment une ind-variété. Dans le cas $n = 2$, on remarque qu'une seule variété suffit.

Introduction

Let k be a field, and let p be a prime number that is invertible in k. The notion of a norm variety was introduced in the study of the Bloch-Kato conjecture. It is a key tool in the proof provided by Rost, Suslin and Voevodsky. The norm variety $X(s)$ of a pure symbol

$$s = (x_1) \cup (x_2) \cup \cdots \cup (x_n) \in H^n(k, \mu_p^{\otimes n}),$$

where the x_i's are elements of $k^\times$, was constructed by Rost (cf. [5] or [7]). The terminology 'norm variety' reflects that it is defined through an inductive process involving the norm of

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finite field extensions of degree p. It has the remarkable property that, if ℓ/k is a field extension, then the restriction of s vanishes in $H^n(\ell, \mu_p^{\otimes n})$ if and only if the ℓ-variety $X(s)_\ell$ has a 0-cycle of degree prime-to-p. It enjoys nice geometric features, which we will not mention here. For $n \geq 3$, norm varieties are, to the knowledge of the authors of this paper, known to exist for pure symbols only.

In this paper, we shall be interested in the following closely related problem. Let A/k be a finite étale group scheme of multiplicative type, that is to say, a finite discrete Γ-module. Consider a class $x \in H^n(k, A)$. Does there exist a *countable* family of smooth k-varieties $(X_i)_{i\in I}$ such that, for every field extension ℓ/k, the presence of a ℓ-point in (at least) one of the X_i's is equivalent to the vanishing of x in $H^n(\ell, A)$? If such a family exists, can it always be endowed with the structure of an ind-variety?

We provide answers to those questions. The main results of the paper are the following:

THEOREM 0.1 (Corollary 4.2 and Corollary 5.8). – *Let A/k be a finite étale group scheme of multiplicative type and let $\alpha \in H^n(k, A)$, where $n \geq 2$ is an integer.*

— *There exists a countable family $(X_i)_{i\in I}$ of smooth geometrically integral k-varieties, such that for any field extension ℓ/k with ℓ infinite, α vanishes in $H^n(\ell, A)$ if and only if $X_i(\ell) \neq \emptyset$ for some i. In addition, there is such a family (X_i) which is an ind-variety.*

— *If $n = 2$, the family (X_i) can be replaced by a single smooth geometrically integral k-variety.*

Note that our main "non-formal" tool, as often (always?) in this context, is Hilbert's Theorem 90.

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1. Notation and definitions

In this paper, k is a field, with a given separable closure k^s. We denote by $\Gamma := \mathrm{Gal}(k^s/k)$ the absolute Galois group. The letters d and n denote two positive integers. We assume d to be invertible in k.

We denote by $\mathcal{M}_d$ the Abelian category of finite $\mathbf{Z}/d\mathbf{Z}$-modules, and by $\mathcal{M}_{\Gamma,d}$ that of finite and discrete d-torsion Γ-modules. The latter is equivalent to the category of finite k-group schemes of multiplicative type, killed by d. We denote this category by $\mathcal{M}_{k,d}$. When no confusion can arise, we will identify these categories without further notice. We have an obvious forgetful functor $\mathcal{M}_{\Gamma,d} \to \mathcal{M}_d$.

1.1. Groups and cohomology.

Let G be a linear algebraic k-group; that is, an affine k-group scheme of finite type. We denote by $H^1(k, G)$ the set of isomorphism classes of G-torsors, for the fppf topology. It coincides with the usual Galois cohomology set if G is smooth. Let $\varphi : H \to G$ be a morphism of linear algebraic k-groups. It induces, for every field extension ℓ/k, a natural map $H^1(\ell, H) \to H^1(\ell, G)$, which we denote by $\varphi_{\ell,*}$.

1.2. Yoneda Extensions.

Let $\mathcal{A}$ be an Abelian category. For all $n \geq 0$, $A, B \in \mathcal{A}$, we denote by $\mathbf{YExt}^n_{\mathcal{A}}(A, B)$ (or $\mathbf{YExt}^n(A, B)$) the (additive) category of Yoneda n-extensions of B by A, and by $\mathrm{YExt}^n_{\mathcal{A}}(A, B)$ (or $\mathrm{YExt}^n(A, B)$) the Abelian group of Yoneda equivalence classes in $\mathbf{YExt}^n(A, B)$. Recall (see [4], Section 2) that an object of $\mathbf{YExt}^n_{\mathcal{A}}(A, B)$ is an exact sequence

$$\mathcal{E} = (0 \to B \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdots \to E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} A \to 0)$$

of objects in $\mathcal{A}$, and morphisms between two such n-extensions of A by B are morphisms of complexes for which the induced morphism from A (resp. B) to itself is the identity map.

Recall also that one says that two n-extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathbf{YExt}^n_{\mathcal{A}}(A, B)$ are equivalent if there exists a third extension $\mathcal{E}$ in $\mathbf{YExt}^n_{\mathcal{A}}(A, B)$ and morphisms of n-extensions $\mathcal{E}_1 \leftarrow \mathcal{E} \to \mathcal{E}_2$. In our setting, this indeed defines an equivalence relation between objects of $\mathbf{YExt}^n_{\mathcal{A}}(A, B)$ (see for instance [4], end of Section 2).

Remark 1.1. – The groups $\mathrm{YExt}^n_{\mathcal{A}}(A, B)$ can also be defined as $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n])$, where $\mathcal{D}(\mathcal{A})$ denotes the derived category of $\mathcal{A}$.

Given $A, B \in \mathcal{M}_d$, we put $\mathbf{YExt}^n_d(A, B) := \mathbf{YExt}^n_{\mathcal{M}_d}(A, B)$. Given $A, B \in \mathcal{M}_{k,d}$, we put $\mathbf{YExt}^n_{k,d}(A, B) := \mathbf{YExt}^n_{\mathcal{M}_{k,d}}(A, B)$.

Remark 1.2. – Let A be a finite discrete Γ-module. Let d be the exponent of A.

The group $\mathrm{YExt}^n_{k,d}(\mathbf{Z}/d\mathbf{Z}, A)$ coincides with the usual Ext-group defined via injective resolutions (see [9], Ch. III, Section 3), and we have a natural isomorphism $\mathrm{YExt}^0_{k,d}(\mathbf{Z}/d\mathbf{Z}, A) \xrightarrow{\sim} A^\Gamma$, therefore there is a canonical isomorphism

$$\mathrm{YExt}^n_{k,d}(\mathbf{Z}/d\mathbf{Z}, A) \xrightarrow{\sim} H^n(\Gamma, A),$$

where $H^n(\Gamma, A)$ denotes the usual n-th cohomology group.

Remark 1.3. – Let ℓ/k be any field extension. For $A, B \in \mathcal{M}_{k,d}$, we have a restriction map

$$\mathrm{Res}_{\ell/\mathrm{k}} : \mathbf{YExt}^n_{k,d}(A, B) \longrightarrow \mathbf{YExt}^n_{\ell,d}(A, B).$$

1.3. Lifting triangles.

Let $\varphi : H \to G$ be a morphism of linear k-algebraic groups. A lifting triangle (relative to φ) is a commutative triangle T:

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ & \underset{H_X}{\searrow} & \downarrow{\scriptstyle G_X} \\ & & X, \end{array}$$

where X is a k-scheme, $Q \longrightarrow X$ (resp. $P \longrightarrow X$) is an H_X-torsor (resp. a G_X-torsor), and where f is an H-equivariant morphism (formula on the functors of points: $f(h.x) = \varphi(h).f(x)$).

Note that such a diagram is equivalent to the data of an isomorphism between the G_X-torsors P and $\varphi_*(Q)$.

The k-scheme X is called the *base* of the lifting triangle T.