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ON THE DYNAMICS OF MINIMAL HOMEOMORPHISMS OF $\mathbb{T}^2$ WHICH ARE NOT PSEUDO-ROTATIONS

BY ALEJANDRO KOCSARD

ABSTRACT. – We prove that any minimal 2-torus homeomorphism which is isotopic to the identity and whose rotation set is not just a point exhibits uniformly bounded rotational deviations on the perpendicular direction to the rotation set. As a consequence of this, we show that any such homeomorphism is topologically mixing and we prove Franks-Misiurewicz conjecture under the assumption of minimality.

RÉSUMÉ. – Soit f un homéomorphisme minimal du tore $\mathbb{T}^2$ qui est isotope à l'identité. Nous montrons que si son ensemble de rotation $\rho(f)$ n'est pas trivial (i.e., il n'est pas un singleton), alors les déviations rotationnelles dans la direction perpendiculaire à l'ensemble de rotation sont uniformément bornées. Par conséquent, nous prouvons qu'un tel homéomorphisme f est topologiquement mélangeant et on donne une démonstration de la conjecture de Franks et Misiurewicz pour homéomorphismes minimaux.

1. Introduction

The study of the dynamics of orientation preserving circle homeomorphisms has a long and well established history that started with the celebrated work of Poincaré [27]. If $f\colon \mathbb{T} = \mathbb{R}/\mathbb{Z} \circlearrowleft$ denotes such a homeomorphism and $\tilde{f}\colon \mathbb{R} \circlearrowleft$ is a lift of f to the universal cover, he showed that there exists a unique $\rho \in \mathbb{R}$, the so called *rotation number* of $\tilde{f}$, such that

$$\frac{\tilde{f}^n(z) - z}{n} \to \rho, \quad \text{as } n \to \infty,\ \forall z \in \mathbb{R},$$

where the convergence is uniform in z. Moreover, in this case a stronger (and very useful, indeed) condition holds: every orbit exhibits *uniformly bounded rotational deviations*, i.e.,

$$\left| \tilde{f}^n(z) - z - n\rho \right| \leq 1, \quad \forall n \in \mathbb{Z},\ \forall z \in \mathbb{R}.$$

In this setting, the homeomorphism f has no periodic orbit if and only if the rotation number is irrational; and any minimal circle homeomorphism is topologically conjugate to a rigid irrational rotation.

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However, in higher dimensions the situation dramatically changes. If $f\colon \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \circlearrowleft$ is a homeomorphism homotopic to the identity and $\tilde{f}\colon \mathbb{R}^d \circlearrowleft$ is a lift of f, then one can define its *rotation set* by

$$\rho(\tilde{f}) := \left\{\rho \in \mathbb{R}^d : \exists n_k \uparrow +\infty,\ z_k \in \mathbb{R}^d,\ \rho = \lim_{k\to+\infty} \frac{\tilde{f}^{n_k}(z_k) - z_k}{n_k}\right\}.$$

This set is always compact and connected, and as we mentioned above, it reduces to a point when $d = 1$. But for $d \geq 2$ some examples with larger rotation sets can be easily constructed.

In the two-dimensional case, which is the main scenario of this work, Misiurewicz and Ziemian showed in [24] that the rotation set is not just connected but convex. So, when $d = 2$ all torus homeomorphisms of the identity isotopy class can be classified according to the geometry of their rotation sets: they can either have non-empty interior, or be a non-degenerate line segment, or be just a point. In the last case, such a homeomorphism is called a *pseudo-rotation*.

Regarding the boundedness of rotational deviations, this property has been shown to be very desirable in the study of the dynamics of pseudo-rotations (see for instance the works of Jäger and collaborators [9, 10, 11]). However, it has been proved in [12] and [15] that, in general, pseudo-rotations do not exhibit bounded rotational deviations in any direction of $\mathbb{R}^2$, i.e., it can hold

$$\sup_{z\in\mathbb{R}^2, n\in\mathbb{Z}} \left\langle \tilde{f}^n(z) - z - n\rho(\tilde{f}), v\right\rangle = +\infty, \quad \forall v \in \mathbb{S}^1.$$

When $\rho(\tilde{f})$ is a (non-degenerate) line segment, of course there exist points with different rotation vectors, so we cannot expect to have any boundedness at all for rotational deviations on the plane. However, in such a case there exists a unit vector $v \in \mathbb{S}^1$ and a real number α such that $\rho(\tilde{f})$ is contained in the line $\{z \in \mathbb{R}^2 : \langle z, v\rangle = \alpha\}$, so one can analyze the boundedness of *rotational v-deviations,* i.e., whether there exist constants $M(z) \in \mathbb{R}$ such that

$$\left|\left\langle \tilde{f}^n(z) - z - n\rho, v\right\rangle\right| = \left|\left\langle \tilde{f}^n(z) - z, v\right\rangle - n\alpha\right| \leq M(z), \quad \forall n \in \mathbb{Z},$$

and any $\rho \in \rho(\tilde{f})$.

Unlike the case of pseudo-rotations, when $\rho(\tilde{f})$ is a non-degenerate line segment in general it is expected to have uniformly bounded rotational v-deviations, i.e., the constant $M(z)$ can be taken independently of z. This result has been already proved by Dávalos [3] in the case where $\rho(\tilde{f})$ has rational slope and intersects $\mathbb{Q}^2$, extending a previous result of Guelman, Koropecki and Tal [7]. In those works periodic orbits of f play a key role.

However, the situation is considerably subtler when dealing with periodic point free homeomorphisms. So far there did not exist any a priori boundedness of rotational deviations of torus homeomorphisms which are not pseudo-rotations and with no periodic points. In fact, it had been conjectured that any periodic point free homeomorphism should be a pseudo-rotation. More precisely, Franks and Misiurewicz had proposed in [5] the following

CONJECTURE 1.1 (Franks-Misiurewicz Conjecture). – *Let $f\colon \mathbb{T}^2 \circlearrowleft$ be a homeomorphism homotopic to the identity and $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ be a lift of f such that $\rho(\tilde{f})$ is a non-degenerate line segment.*

Then, the following dichotomy holds:

(i) *either* $\rho(\tilde{f})$ *has irrational slope and one of its extreme points belongs to* $\mathbb{Q}^2$*;*

(ii) *or* $\rho(\tilde{f})$ *has rational slope and contains infinitely many rational points.*

Recently Avila has announced the existence of a minimal smooth diffeomorphism whose rotation set is an irrational slope segment containing no rational point, providing in this way a counter-example to the first case of Franks-Misiurewicz Conjecture. On the other hand, Le Calvez and Tal have proved in [21] that if $\rho(\tilde{f})$ has irrational slope and contains a rational point, then this point is an extreme one.

The second case of Conjecture 1.1 remains open, i.e., whether there exists a homeomorphism f such that $\rho(\tilde{f})$ has rational slope and $\rho(\tilde{f}) \cap \mathbb{Q}^2 = \emptyset$, and in fact this is one of the main motivations of our work.

The main result of this paper is the following *a priori boundedness* for rotational deviations of minimal homeomorphisms:

THEOREM A. – *Let* $f\colon \mathbb{T}^2 \circlearrowleft$ *be a minimal homeomorphism homotopic to the identity which is not a pseudo-rotation. Then there exists a unit vector* $v \in \mathbb{R}^2$ *and a real number* $M > 0$ *such that for any lift* $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$*, there is* $\alpha \in \mathbb{R}$ *so that*

$$\left|\left\langle \tilde{f}^n(z) - z, v\right\rangle - n\alpha\right| \leq M, \quad \forall z \in \mathbb{R}^2,\ \forall n \in \mathbb{Z}. \tag{1}$$

As a consequence of Theorem A and a recent result due to Koropecki, Passeggi and Sambarino [14], we get a proof of the second case of Franks-Misiurewicz Conjecture (Conjecture 1.1) under minimality assumption. More precisely we get the following:

THEOREM B. – *There is no minimal homeomorphism of* $\mathbb{T}^2$ *in the identity isotopy class such that its rotation set is a non-degenerate rational slope segment.*

As a consequence of Theorem B, some results of [13] and a recent generalization of a theorem of Kwapisz [18] due to Beguin, Crovisier and Le Roux [2], we have the following

THEOREM C. – *If* $f\colon \mathbb{T}^2 \circlearrowleft$ *is a minimal homeomorphism homotopic to the identity and is not a pseudo-rotation, then* f *is topologically mixing.*

Moreover, in such a case the rotation set of f *is a non-degenerate irrational slope line segment and its supporting line does not contain any point of* $\mathbb{Q}^2$.

1.1. Strategy of the proof of Theorem A

Theorem A is certainly the most important result of the paper and its proof is rather long and technical. So, for the sake of readability, here we summarize the main steps of the proof in a rather informal way.

We proceed by contradiction. First of all one can observe that there is no loss of generality assuming the rotation set $\rho(\tilde{f})$ is transversal to the horizontal axes, i.e., it intersects the upper and lower horizontal semi-planes (see Propositions 2.5 and 2.16 for details). This means there exist points with a positive asymptotic vertical mean speed and others with a negative one.