

*quatrième série - tome 55    fascicule 1    janvier-février 2022*

*ANNALES  
SCIENTIFIQUES  
de  
L'ÉCOLE  
NORMALE  
SUPÉRIEURE*

Harold BLUM & Yuchen LIU

*Openness of uniform  $K$ -stability in families of  $\mathbb{Q}$ -Fano varieties*

---

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

---

Publiées avec le concours du Centre National de la Recherche Scientifique

**Responsable du comité de rédaction / *Editor-in-chief***

YVES DE CORNULIER

**Publication fondée en 1864 par Louis Pasteur**

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE  
de 1883 à 1888 par H. DEBRAY  
de 1889 à 1900 par C. HERMITE  
de 1901 à 1917 par G. DARBOUX  
de 1918 à 1941 par É. PICARD  
de 1942 à 1967 par P. MONTEL

**Comité de rédaction au 1<sup>er</sup> octobre 2021**

S. CANTAT      G. GIACOMIN  
G. CARRON      D. HÄFNER  
Y. CORNULIER   D. HARARI  
F. DÉGLISE      C. IMBERT  
A. DUCROS      S. MOREL  
B. FAYAD      P. SHAN

**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,  
45, rue d'Ulm, 75230 Paris Cedex 05, France.  
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.  
Email : [annaes@ens.fr](mailto:annaes@ens.fr)

---

**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France  
Case 916 - Luminy  
13288 Marseille Cedex 09  
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51  
Email : [abonnements@smf.emath.fr](mailto:abonnements@smf.emath.fr)

**Tarifs**

Abonnement électronique : 441 euros.  
Abonnement avec supplément papier :  
Europe : 619 €. Hors Europe : 698 € (\$ 985). Vente au numéro : 77 €.

---

© 2022 Société Mathématique de France, Paris

En application de la loi du 1<sup>er</sup> juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

*All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.*

---

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand  
Périodicité : 6 n<sup>os</sup> / an

# OPENNESS OF UNIFORM K-STABILITY IN FAMILIES OF $\mathbb{Q}$ -FANO VARIETIES

BY HAROLD BLUM AND YUCHEN LIU

---

**ABSTRACT.** – We show that uniform K-stability is a Zariski open condition in  $\mathbb{Q}$ -Gorenstein families of  $\mathbb{Q}$ -Fano varieties. To prove this result, we consider the behavior of the stability threshold in families. The stability threshold (also known as the delta-invariant) is a recently introduced invariant that is known to detect the K-semistability and uniform K-stability of a  $\mathbb{Q}$ -Fano variety. We show that the stability threshold is lower semicontinuous in families and provides an interpretation of the invariant in terms of the K-stability of log pairs.

**RÉSUMÉ.** – Nous démontrons que la K-stabilité uniforme est une condition ouverte pour la topologie de Zariski dans une famille  $\mathbb{Q}$ -Gorenstein de variétés algébriques  $\mathbb{Q}$ -Fano. Pour prouver ce résultat, nous considérons le seuil de stabilité en familles. Le seuil de stabilité (également appelé l'invariant delta) est un invariant qui détecte la K-semistabilité et la K-stabilité uniforme d'une variété algébrique  $\mathbb{Q}$ -Fano. Nous démontrons que le seuil de stabilité est semi-continu inférieurement en familles et fournissons une interprétation de cet invariant en termes de K-stabilité des paires log.

*Throughout, we work over a characteristic zero algebraically closed field  $k$ .*

## 1. Introduction

In this article, we consider the behavior of K-stability in families of  $\mathbb{Q}$ -Fano varieties. Recall that K-stability is an algebraic notion introduced by Tian [53] and later reformulated by Donaldson [21] to detect the existence of certain canonical metrics on complex projective varieties. In the special case of complex  $\mathbb{Q}$ -Fano varieties, the Yau-Tian-Donaldson conjecture states that a complex  $\mathbb{Q}$ -Fano variety is K-polystable iff it admits a Kähler-Einstein metric. (By a  $\mathbb{Q}$ -Fano variety, we mean a projective variety that has at worst klt singularities and anti-ample canonical divisor.) For smooth complex Fano varieties, this conjecture was recently settled in the work of Chen-Donaldson-Sun and Tian [16, 54] (see also [19, 17, 2]).

Another motivation for understanding the K-stability of  $\mathbb{Q}$ -Fano varieties is to construct compact moduli spaces for such varieties. It is expected that there is a proper good moduli space parametrizing K-polystable  $\mathbb{Q}$ -Fano varieties of fixed dimension and volume. For

smoothable  $\mathbb{Q}$ -Fano varieties, such a moduli space is known to exist [40] (see also [48, 43]). A key step in constructing the moduli space of  $\mathbf{K}$ -polystable Fano varieties is verifying the Zariski openness of  $\mathbf{K}$ -semistability. Towards this goal, we prove

**THEOREM A.** – *If  $\pi : X \rightarrow T$  is a projective flat family of varieties such that  $T$  is normal,  $\pi$  has normal connected fibers, and  $-K_{X/T}$  is  $\mathbb{Q}$ -Cartier and  $\pi$ -ample, then*

- (1)  $\{t \in T \mid X_{\bar{t}} \text{ is uniformly } \mathbf{K}\text{-stable}\}$  is a Zariski open subset of  $T$ , and
- (2)  $\{t \in T \mid X_{\bar{t}} \text{ is } \mathbf{K}\text{-semistable}\}$  is a countable intersection of Zariski open subsets of  $T$ .

The notion of uniform  $\mathbf{K}$ -stability is a strengthening of  $\mathbf{K}$ -stability introduced in [11, 20]. In [2], it was shown that a smooth Fano variety  $X$  with discrete automorphism group is uniformly  $\mathbf{K}$ -stable iff there exists a Kähler-Einstein metric on  $X$ . The latter equivalence was later extended to  $\mathbb{Q}$ -Fano varieties with discrete automorphism group in [55].  $\mathbf{K}$ -semistability is strictly weaker than  $\mathbf{K}$ -(poly)stability and corresponds to being almost Kähler-Einstein [37, 2].

In [7], the first author and Xu show that the moduli functor of uniformly  $\mathbf{K}$ -stable  $\mathbb{Q}$ -Fano varieties of fixed volume and dimension is represented by a separated Deligne-Mumford stack, which has a coarse moduli space that is a separated algebraic space. The proof of the result combines Theorem A.1 with a boundedness statement in [28] (that uses ideas from [3]) and a separatedness statement in [7].

For smooth families of Fano varieties, Theorem A is not new. Indeed, for a smooth family of complex Fano varieties with discrete automorphism group, the  $\mathbf{K}$ -stable locus is Zariski open by [44, 23]. In [40], it was shown that the  $\mathbf{K}$ -semistable locus is Zariski open in families of smoothable  $\mathbb{Q}$ -Fano varieties. These results all rely on deep analytic tools developed in [16, 54].

Unlike the previous results, our proof of Theorem A is purely algebraic. (A different algebraic proof of Theorem A.2 was also given in [6] using a characterization of  $\mathbf{K}$ -semistability in terms of the normalized volume of the affine cone over a  $\mathbb{Q}$ -Fano variety [36, 38, 41].) Furthermore, the result holds for all  $\mathbb{Q}$ -Fano varieties, including those that are not smooth(able), and also log Fano pairs. The argument relies on new tools for characterizing the uniform  $\mathbf{K}$ -stability and  $\mathbf{K}$ -semistability of Fano varieties [11, 36, 24, 25, 5].

More precisely, our approach to proving Theorem A is through understanding the behavior of the *stability threshold* (also known as  $\delta$ -invariant or *basis log canonical threshold*) in families. We recall the definition of this new invariant.

Let  $X$  be a projective klt variety and  $L$  an ample Cartier divisor on  $X$ . Set

$$|L|_{\mathbb{Q}} := \{D \in \text{Div}(X)_{\mathbb{Q}} \mid D \geq 0 \text{ and } mD \sim mL \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

Following [25], we say that  $D \in |L|_{\mathbb{Q}}$  is an  $m$ -basis type divisor of  $L$  if there exists a basis  $\{s_1, \dots, s_{N_m}\}$  of  $H^0(X, \mathcal{O}_X(mL))$  such that

$$D = \frac{1}{mN_m} (\{s_1 = 0\} + \dots + \{s_{N_m} = 0\}).$$

For  $m \in M(L) := \{m \mid h^0(X, \mathcal{O}_X(mL)) \neq 0\}$ , set

$$\delta_m(X; L) := \inf_{D \text{ } m\text{-basis type}} \text{lct}(X; D),$$

where  $\text{lct}(X; D)$  denotes the log canonical threshold of  $D$ . The stability threshold of  $L$  is

$$\delta(X; L) := \limsup_{M(L) \ni m \rightarrow \infty} \delta_m(X; L).$$

In fact, the above limsup is a limit by [5]. If  $X$  is a  $\mathbb{Q}$ -Fano variety, we set

$$\delta(X) := r\delta(X; -rK_X),$$

where  $r \in \mathbb{Z}_{>0}$  is such that  $-rK_X$  is Cartier. (The definition is independent of the choice of  $r$ .)

The stability threshold is closely related to the *global log canonical threshold of  $L$* , which is an algebraic version of Tian's  $\alpha$ -invariant. Recall that the global log canonical threshold of  $L$  is

$$\alpha(X; L) := \inf_{D \in |L|_{\mathbb{Q}}} \text{lct}(X; D).$$

The two thresholds satisfy

$$\frac{n+1}{n}\alpha(X; L) \leq \delta(X; L) \leq (n+1)\alpha(X; L),$$

where  $n = \dim(X)$ .

The stability threshold was introduced in the  $\mathbb{Q}$ -Fano case by K. Fujita and Y. Odaka to characterize the K-stability of  $\mathbb{Q}$ -Fano varieties [25]. More generally, the invariant coincides with an invariant suggested by R. Berman and defined in [12]. Using the valuative criterion for K-stability of K. Fujita and C. Li [24, 36], it was shown that the invariant characterizes certain K-stability notions.

**THEOREM 1.1** ([25, 5]). – *Let  $X$  be a  $\mathbb{Q}$ -Fano variety.*

- (1)  *$X$  is uniformly K-stable iff  $\delta(X) > 1$ .*
- (2)  *$X$  is K-semistable iff  $\delta(X) \geq 1$ .*

In light of the previous statement, Theorem A is a consequence of the following result.

**THEOREM B.** – *Let  $\pi : X \rightarrow T$  be a projective flat family of varieties and  $L$  a  $\pi$ -ample Cartier divisor on  $X$ . Assume  $T$  is normal,  $X_t$  is a klt variety for all  $t \in T$ , and  $K_{X/T}$  is  $\mathbb{Q}$ -Cartier. Then, the two functions  $T \rightarrow \mathbb{R}$  defined by*

$$T \ni t \mapsto \delta(X_{\bar{t}}; L_{\bar{t}}) \quad \text{and} \quad T \ni t \mapsto \alpha(X_{\bar{t}}; L_{\bar{t}})$$

*are lower semicontinuous.*

In the above theorem,  $(X_{\bar{t}}; L_{\bar{t}})$  denotes the restriction of  $(X, L)$  to the geometric fiber over  $t$ . As explained in [18, Remark 4.15], the above result would not hold with “ $\delta(X_{\bar{t}}; L_{\bar{t}})$ ” replaced by “ $\delta(X_t; L_t)$ .”

Let us note the main limitation of Theorem B. While the statement implies that  $\{t \in T \mid \delta(X_{\bar{t}}; L_{\bar{t}}) > a\}$  is open for each  $a \in \mathbb{R}_{\geq 0}$ , it does not imply the fact that  $t \mapsto \delta(X_{\bar{t}}; L_{\bar{t}})$  takes finitely many values. Hence, we are unable to prove  $\{t \in T \mid \delta(X_{\bar{t}}; L_{\bar{t}}) \geq a\}$  is open and cannot verify the openness of K-semistability in families of  $\mathbb{Q}$ -Fano varieties. The openness of K-semistability is an immediate consequence of Theorem B and the following conjecture (see [6, Conjecture 2] for a local analogue).