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PERIODS OF AUTOMORPHIC FORMS OVER REDUCTIVE SUBGROUPS

BY MICHAŁ ZYDOR

ABSTRACT. – We present a regularization procedure of period integrals of automorphic forms on a group G over an arbitrary reductive subgroup $G' \subset G$. As a consequence we obtain an explicit $G'(\mathbb{A})$-invariant functional on the space of automorphic forms on G whose exponents avoid certain prescribed hyperplanes. We also provide a necessary and sufficient condition for convergence of period integrals of automorphic forms in terms of their exponents.

RÉSUMÉ. – Une procédure de régularisation de périodes de formes automorphes sur un groupe réductif G selon un sous-groupe réductif G' quelconque est décrite. Comme une conséquence on obtient une fonctionnelle $G'(\mathbb{A})$-invariant sur l'espace de formes automorphes sur G dont les exposants évitent certains hyperplans prescrits. On fournit également un critère nécessaire et suffisant pour la convergence absolue de telles périodes en fonction des exposants de formes automorphes.

Introduction

0.1. Our result

Let G be a connected reductive algebraic group and $G' \subset G$ a subgroup, both defined over a number field F. An automorphic period is the integral

$$\mathcal{P}(\phi) := \int_{G'(F)\backslash G'(\mathbb{A})} \phi(g)\, dg, \quad \phi \in \mathcal{A}(G),$$

where $\mathcal{A}(G)$ is the space of automorphic forms on G.

Periods appear in many areas related to number theory. They often pertain to special values of automorphic L-functions and can also be used to study automorphic representations.

This article addresses the question of convergence of period integrals. It was proven in [5] that automorphic periods converge for cuspidal automorphic representations whenever G/G' is a quasi-affine variety. However, unless the automorphic quotient $G'(F)\backslash G'(\mathbb{A})$ is compact, the period integral is ill defined on the space of all automorphic forms $\mathcal{A}(G)$.

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Nevertheless, it was first observed by Zagier [49], that in the realm of automorphic forms, divergent integrals can be regularized, providing a meaningful extension of the functional $\mathcal{P}$, defined initially only for cusp forms, to the space $\mathcal{A}(G)$. Zagier's work concerned the case of the scalar product on GL_2 and was further reinterpreted by Casselman [7]. However, it is the subsequent work of Jacquet-Lapid-Rogawski [22] that provided a framework that inspired our work.

We consider the case where G' is a connected reductive subgroup. Let us describe briefly our approach. We define, what's commonly called a mixed truncation operator

$$\Lambda^T : \mathcal{A}(G) \to \text{ rapidly decreasing functions on } G'(F)\backslash G'(\mathbb{A}). \tag{0.1}$$

The operator is called mixed as opposed to the standard one used by Langlands and Arthur [2]. Here, T is a parameter in an $\mathbb{R}$-vector space of dimension equal to the split rank of G'. We obtain thus a family of operators on $\mathcal{A}(G)$ that send it to the space of rapidly decreasing functions on $G'(F)\backslash G'(\mathbb{A})$. Once such operator is defined and its properties established, cf. Section 3, we can consider the integral

$$\mathcal{P}^T(\phi) := \int_{G'(F)\backslash G'(\mathbb{A})} \Lambda^T \phi(g)\, dg, \quad \phi \in \mathcal{A}(G).$$

The insight of [22], is that such an integral, even though not canonically defined, has a canonically defined constant term in the parameter T, at least for almost all automorphic forms.

One defines then, the so called regularized period

$$\mathcal{P}(\phi) := \text{ constant term of } T \mapsto \mathcal{P}^T(\phi).$$

For $x \in G(\mathbb{A})$ let $\phi_x(y) = \phi(yx^{-1})$. The main result of the paper can be stated then as follows.

THEOREM 0.1 (cf. Theorem 4.1). – *For ϕ in an appropriately defined subspace $\mathcal{A}(G)^*$ of $\mathcal{A}(G)$ we have*

$$\mathcal{P}(\phi_x) = \mathcal{P}(\phi), \quad \forall\, x \in G'(\mathbb{A})$$

and $\mathcal{P}(\phi)$ is independent of any choices.

Moreover, Theorem 4.5 provides a necessary and sufficient condition for period integrals of automorphic forms to converge. The article considers in fact a slightly more general setting where an automorphic character on the group $G'(\mathbb{A})$ is included. In Paragraph 4.6 we briefly explain how to extend the regularization to the case where G' is not connected.

0.2. On the proofs

The core of the proof lies in establishing the rapid decay property of Λ^T. In essence, we follow the approach of Arthur [2]. We hope that the notation we employ makes this connection apparent. What allows us to follow Arthur's approach, and also the one of Jacquet-Lapid-Rogawski is a careful definition of the operator Λ^T. Here's the definition

$$\Lambda^T \phi(x) = \sum_{P \in \mathcal{F}^G(P_0')} \varepsilon_P^G \sum_{\delta \in (P\cap G')\backslash G'(F)} \widehat{\tau}_P(H_{0'}(\delta x) - T)\phi_P(\delta x). \tag{0.2}$$

It should come as no surprise that Λ^T is defined as a certain alternating sum (we have $\varepsilon_P^G = \pm 1$) of constant terms of ϕ (ϕ_P is the constant term of ϕ with respect to the parabolic subgroup P) truncated (via $\widehat{\tau}_P$) and summed to make everything $G'(F)$-invariant. The group P'_0 is a fixed minimal parabolic subgroup of G'. The set $\mathcal{F}^G(P'_0)$ is then defined as the set of parabolic subgroups of G that can be defined via a cocharacter $\lambda : \mathbb{G}_m \to G$ that has values in G' and regarded as a homomorphism $\mathbb{G}_m \to G'$ defines a parabolic subgroup of G' containing P'_0. In particular, the definition ensures that $P \cap G'$ contains P'_0 for $P \in \mathcal{F}^G(P'_0)$. The function $H_{0'}$ is the Harish-Chandra function on $G'(\mathbb{A})$ taking values in the space $\mathfrak{a}_{0'} = \mathrm{Hom}_F(\mathbb{G}_m, A'_0) \otimes_{\mathbb{Z}} \mathbb{R}$ where $A'_0 \subset P'_0$ is a fixed F-split torus of G'. Finally, $\widehat{\tau}_P$ is the characteristic function of the dual cone of the cone $\mathfrak{a}_{0'} \cap \mathfrak{a}_P^+$, $\mathfrak{a}_P^+$ being the standard positive chamber associated to P (and a fixed F-split torus of P determined by A'_0).

The Section 1 studies the various truncation functions, such as $\widehat{\tau}_P$ above. We took an approach of the theory of polyhedral cones. Let us cite [4] for an excellent reference to this theory. It turns out that this framework, albeit a little unorthodox in this setting, meets perfectly the needs of a regularization procedure. In the classical work of Arthur, there are two principal combinatorial constructions—these are the functions Γ and σ. Both are defined as certain alternating sums of characteristic functions of cones. In [50], H. Zheng and the author study the generalization of the Γ function. This work is recalled in Paragraph 1.2. The generalization of the σ-function is studied in Paragraph 1.3.

Once the combinatorial properties of general truncation functions are established in Section 1, we start working in the setting of reductive groups. In Section 2 we set up the notation and study the reduction theory. We do not really prove anything new about the reduction theory of algebraic groups. We mostly recall some classical statements, at times prove minor improvements. It is in Section 3 that we properly introduce the relative setting $G' \subset G$. We define the truncation operator Λ^T and prove some typical combinatorial properties it satisfies. The core result is the rapid decay property (0.1) of Theorem 3.9. Before we can prove it however, we prove the “relative decomposition of 1” result in Proposition 3.5. We provide some pictures of truncated cones throughout the text to provide some insight into our constructions. The final Section 4 reaps the profits of the work performed in the former sections. We define the period as explained above and verify briefly its properties in Theorem 4.1. Paragraph 4.7 spells out our construction in the case of cuspidal Eisenstein series associated to maximal parabolic subgroups of G. Finally, in the last Paragraph 4.8, we prove Theorem 4.5 which provides a definite criterion for convergence of periods of automorphic forms over reductive subgroups.

0.3. Example: symmetric periods

Let θ be an involution on G and suppose $G' = G^\theta$. Fix A_0 a maximal F-split and θ-stable torus of G such that $A'_0 = A_0^\theta$ is a maximal F-split torus of G' (existence is proven in [14]). We fix also P'_0 a minimal parabolic subgroup of G' containing A'_0. We have then $\mathfrak{a}_{0'} = \mathfrak{a}_0^\theta$ and the set $\mathcal{F}^G(P'_0)$ in (0.2) is the set of parabolic subgroups of G that are θ-stable and whose intersection with G' contains P'_0.