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## *A stochastic approach to counting problems*

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# A STOCHASTIC APPROACH TO COUNTING PROBLEMS

BY Adrien BOULANGER

ABSTRACT. – We study orbital functions associated to finitely generated geometrically infinite Kleinian groups acting on the hyperbolic space $\mathbb{H}^3$, developing a new method based on the use of the Brownian motion. On the way, we give some estimates of the orbital function associated to nilpotent covers of compact hyperbolic manifolds, partially answering a question asked by M. Pollicott to the author.

RÉSUMÉ. – Nous étudions les fonctions orbitales associées aux groupes kleiniens finiment engendrés géométriquement infinis agissant sur l'espace hyperbolique $\mathbb{H}^3$. Pour cela, nous développons une méthode basée sur l'utilisation du noyau de la chaleur. La méthode donne également des estimées sur les fonctions orbitales associées aux revêtements nilpotents de 3-variétés hyperboliques compactes, ce qui répond partiellement à une question posée à l'auteur par M. Pollicott.

## 1. Introduction

*Historical background.* – Given a group $\Gamma$ acting properly and discontinuously on a metric space $(X, d)$, we define the *orbital function* as follows

$$N_\Gamma(x, y, \rho) := \sharp\{\, \gamma \in \Gamma,\ d(x, \gamma \cdot y) \leq \rho \,\},$$

where $x, y \in X$ and $\rho > 0$. The growth of orbital functions of groups acting on various types of hyperbolic spaces have been extensively studied since the 50s. Two main approaches were developed in this setting. The oldest one, due to Huber [21] following Delsarte (see [1]), relies on Selberg's pre-trace formula and was designed to study orbital functions in the setting of groups acting cocompactly on the hyperbolic space $\mathbb{H}^2$. This line of work was generalized later on by Selberg (unpublished), Patterson [32] and Lax-Phillips [25], among many others. Another way to get estimates of the orbital function growth comes from Margulis' seminal PhD thesis, which exhibits a strong relation between this problem and the mixing of the

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geodesic flow on the quotient space ${}^{X}\!/_{\Gamma}$. This second approach has the benefit to give results in the case of variable curvature as well. This idea has been widely extended by numerous authors since, dropping most of Margulis' assumptions required in his work [43] [35]. For general references, one can recommend Babillot's survey [1] on counting problems, Section 2 of Eskin-McMullen's article [16] where Margulis' strategy is well explained and the author's PhD dissertation [4, Chapitre 1]. The following theorem is, generality-wise, the most advanced of the theory.

THEOREM 1.1 ([35]). – *Let $\Gamma$ be a group acting by isometries, properly and discontinuously on a connected, simply connected complete manifold $X$ of sectional curvature uniformly bounded above by some negative constant such that*

- *the length spectrum of the quotient space is non arithmetic;*
- *the unitary tangent bundle of the quotient space ${}^{X}\!/_{\Gamma}$ admits a finite Bowen-Margulis-Sullivan measure.*

*Then, for any points $x, y \in X$ there exists a constant $C$ such that*

$$N_\Gamma(x, y, \rho) \underset{\rho\to\infty}{\sim} C\, e^{\delta_\Gamma \rho},$$

*where $\delta_\Gamma$ is the critical exponent of the group $\Gamma$.*

Recall that the critical exponent is, in full generality, defined as

$$\delta_\Gamma := \limsup_{\rho\to\infty} \frac{\ln(N_\Gamma(x, y, \rho))}{\rho},$$

which was proven to be a limit in [34]. The above theorem is weaker than the one stated in [35] but the author did not want to burden this article with definitions regarding the CAT$(-1)$ setting for which an analogous version holds. Let us note that the two assumptions of Theorem 1.1 are automatically satisfied when $\Gamma$ acts convex-compactly on a hyperbolic space of any dimension, see [20, Proposition 3] for the assumption made on the non-arithmeticity of the length spectrum.

Roblin also showed, under the same assumptions of those of Theorem 1.1, that if the quotient space ${}^{X}\!/_{\Gamma}$ does not admit any finite Bowen-Margulis-Sullivan measure, one has

$$N_\Gamma(x, y, \rho) = o\left(e^{\delta_\Gamma \rho}\right),$$

for any pair of points $x, y \in X$ when $\rho \to \infty$. Nowadays, the goal is to explicit the asymptotic in this case. The first one is due to Epstein [14, 15]. The author found, by an approach based on the spectral theory, an asymptotic for orbital functions of Abelian covers of finite volume hyperbolic manifolds of dimension 2 and 3. Pollicott and Sharp in [33] investigated this problem with a dynamical method. They found an asymptotic of the orbital function associated with Abelian covers of hyperbolic manifolds in any dimension but under the extra assumption that the quotient manifold is compact. The second one is due to Vidotto [42] and deals with an example in variable curvature. The dynamical method of [33] and [42] relies on a finite measure property hidden somewhere: in [33] the author's method uses a strong mixing property of the geodesic flow of the compact underlying manifold whereas in [42] the geodesic flow still admits a coding by a sub-shift of finite type.

*The strategy and the setting*. – In this article we introduce a method to study the growth of orbital functions of groups acting on the real hyperbolic space $\mathbb{H}^3$ of dimension three, based on the use of the Brownian motion in the spirit of Sullivan's work [41]. Our motivation for introducing the Brownian motion in this setting is to compensate the lack of known randomness of the geodesic flow—preventing to adapt Margulis' method—when there is no finite Bowen-Margulis-Sullivan measure. Replacing the geodesic flow by the Brownian motion gives a larger range of applications, the later being random on its own. The key facts allowing us to relate its dynamical behavior to the orbital function is the *strong homogeneity* of the hyperbolic space and the so called *drift property* of the Brownian motion. We refer to the author's PhD dissertation [4] for more details.

More precisely, this article aims at investigating the link between Brownian motion and orbital functions through examples. We call *Kleinian group* any torsion free subgroup $\Gamma$ of the orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$ acting properly and discontinuously or, equivalently, the fundamental group of a hyperbolic manifold of dimension 3. Regarding the counting problem for finitely generated Fuchsian groups, analogous to the last but acting on $\mathbb{H}^2$, there is not so much to be said since all the quotients $\mathbb{H}^2/\Gamma$ turned out to carry a finite Bowen-Margulis-Sullivan measure [13], so that Theorem 1.1 readily applies in this case. Therefore, one would like to picture what is the situation in dimension 3. We will call a *finitely generated* Kleinian group $\Gamma$ *degenerate* when the quotient manifold $M_\Gamma := \mathbb{H}^3/\Gamma$ does not carry any finite Bowen-Margulis-Sullivan measure. Note that we require $\Gamma$ to be finitely generated in order to qualify it degenerate. By extension, we also call degenerate the manifold $M_\Gamma$. These manifolds are central objects in some of the most famous theorems around the theory of finitely generated Kleinian groups, such as Thurston's hyperbolisation theorem (see [28] or [31]), Ahlfors' conjecture, corollary [7] of Marden's tameness conjecture (see [8] and the reference therein) or the ending lamination conjecture [5], all theorems now.

Surprisingly, very little is known about the behavior of their orbital functions except that they have critical exponent 2 [27], which is to say that its exponential growing rate is the one of the volume growth of $\mathbb{H}^3$. The only special case already settled being as a corollary of [33], giving an asymptotic to the orbital function for surface groups associated to $\mathbb{Z}$-covers of compact 3-manifolds, included in M. Pollicott and R. Sharp's theorem as Abelian covers of compact manifolds.

*Description of the results*. – Our two main theorems can be stated as follows. We refer to Subsection 4.1 for the definition and the description of ends of hyperbolic manifolds involved in the statements.

THEOREM 1.2. – *Let $\Gamma$ be a degenerate Kleinian group such that the manifold $M_\Gamma$ has positive injectivity radius and has both degenerate and geometrically finite ends. Then, for any $x, y \in \mathbb{H}^3$ there are constant $C > 0$ and $\rho_0 > 0$ such that for any $\rho > \rho_0$ we have*

$$N_\Gamma(x, y, \rho) \leq C\, \frac{e^{2\rho}}{\rho}.$$

Under some extra assumption, one can improve the above theorem into