Séminaire BOURBAKI Octobre 2012
65e année, 2012-2013, no 1062, p. 115 à 148

THE BAUM-CONNES CONJECTURE WITH COEFFICIENTS FOR WORD-HYPERBOLIC GROUPS [after Vincent Lafforgue]

by **Michael PUSCHNIGG**

INTRODUCTION

In a recent breakthrough V. Lafforgue verified the Baum-Connes conjecture with coefficients for all word-hyperbolic groups [41]. This is spectacular progress since it provides the first examples of groups with Kazhdan's property (T) satisfying the conjecture with coefficients [(1)]. Lafforgue's proof is elementary, but of impressive complexity.

In fact, the Baum-Connes conjecture with coefficients is known to be false in general. The first counterexamples were obtained by N. Higson, V. Lafforgue, and G. Skandalis [24] for certain classes of Gromov's random groups [19]. (Note that Gromov's groups are nothing but inductive limits of word-hyperbolic groups!)

Already in the early eighties, A. Connes emphasized that Kazhdan's property (T), which means that the trivial representation of a locally compact group is separated from all other unitary representations, might be a serious obstruction to the Baum-Connes conjecture. The only previously known approach, due to Kasparov [32], demands the construction of a homotopy among unitary representations between the regular and the trivial representation, which cannot exist for non-compact groups with Property (T). This led to a search for such homotopies among larger classes of representations [26, 36, 41]. V. Lafforgue [38] introduces the notion of group representations of weak exponential growth. He shows that the trivial representation is not isolated among such representations for hyperbolic groups which opens the way to his proof of the Baum-Connes conjecture with coefficients. For higher rank groups and lattices however, a corresponding version of Property (T) continues to hold [38, 39]. This leads to interesting applications in graph theory and rigidity theory [39] and

1. A proof for the Property (T) groups $Sp(n, 1)$ has been announced earlier by P. Julg in [28].

makes it hard to believe that the Baum-Connes conjecture (at least in the case with coefficients) might be proved for higher rank lattices by the established methods [40].

In Section 1, we review index theory and formulate the Baum-Connes conjecture as a deep and far reaching generalization of the Atiyah-Singer index theorem. The tools which are used to approach the conjecture are presented in Section 2: Kasparov's bivariant K-theory [30, 32], and his construction of "γ-elements". Section 3 collects the present knowledge about the Baum-Connes conjecture. In particular, we explain the counterexamples of Higson, Lafforgue, and Skandalis. Section 4 deals with Lafforgue's work on generalizations of Kazhdan's property (T). We discuss his results on his Strengthened Property (T) for higher rank groups and lattices and give an account of their proofs. The applications of his work in graph theory and rigidity theory are mentioned as well. In Section 5 we finally outline V. Lafforgue's proof of the Baum-Connes conjecture with coefficients for word-hyperbolic groups.

Acknowledgements. — I thank Vincent Lafforgue very heartily for his help and advices during the preparation of this manuscript. It is a pleasure to thank Nigel Higson, Georges Skandalis, and Guoliang Yu for their explanations and constructive remarks.

1. THE BAUM-CONNES CONJECTURE

1.1. Index theory

Consider a linear elliptic differential operator D on a smooth compact manifold M. Its **analytical index** is defined as

$$\text{Ind}_a(D) = \dim(\text{Ker}\, D) - \dim(\text{CoKer}\, D) \in \mathbb{Z}. \tag{1.1}$$

The analytical index is invariant under perturbations of the elliptic operator and turns out to be calculable by topological means. In fact, it only depends on the class

$$[\sigma_{\text{pr}}(D)] \in K^0(T^*M) \tag{1.2}$$

of the principal symbol of D. Here T^*M is the total space of the cotangent bundle of M and K^* denotes (compactly supported) topological K-theory [3]. (The latter K-group can actually be identified with the set of homotopy classes of pseudo-elliptic symbols.) The topological K-theory of Atiyah-Hirzebruch is a generalized oriented cohomology theory in the sense of algebraic topology. K-oriented manifolds, for example the total space of the cotangent bundle T^*M of a compact manifold M, therefore satisfy a K-theoretic version of Poincaré duality. The image of the symbol class under

$$K^0(T^*M) \xrightarrow{PD} K_0(M) \xrightarrow{p_*} K_0(pt) = \mathbb{Z}, \tag{1.3}$$

$p : M \to pt$ the constant map, is called the *topological index* $\text{Ind}_t(D)$ of D.

Suppose now that a compact Lie group H acts smoothly on M, leaving D invariant. Then kernel and cokernel of D become finite-dimensional H-modules and one may define the equivariant analytical index of D

$$\mathrm{Ind}_a(D) = [\mathrm{Ker}\, D] - [\mathrm{CoKer}\, D] \in R(H), \tag{1.4}$$

as element of the representation ring $R(H)$. The equivariant topological index can be defined in a similar way as before as an element of the equivariant K-homology group $K_0^H(pt)$ of a point. There is a tautological isomorphism

$$\mu : K_0^H(pt) \xrightarrow{\simeq} R(H) \tag{1.5}$$

which allows to view both equivariant indices as virtual finite dimensional representations of H. The *Atiyah-Singer Index Theorem* reads then as follows:

Theorem 1.1 ([3, (6.7)]). — *The analytical index and the topological index coincide as homomorphisms $K_H(T^*M) \to R(H)$.*

1.2. Higher index theory

Kasparov [32] and Baum-Connes [7, 8] claim that a similar index theorem holds in the following much more general setting:

- G is an arbitrary locally compact group,
- M is a smooth manifold equipped with a proper and cocompact G-action,
- D is a G-invariant linear elliptic differential operator on M.

Note that the condition on the action of G implies that M is non-compact if G is. In particular, D cannot be Fredholm in any naive sense for non-compact G. Thus completely new ideas are needed to give a meaning to an "analytical index".

Assume that the locally compact group G acts smoothly and properly on the manifold M. Then there exists a G-invariant smooth positive measure *dvol* on M. The corresponding Sobolev spaces become G-Hilbert spaces, which appear as subrepresentations of a (countable) multiple of the (left)-regular representation on $L^2(G)$.

Definition 1.2. — *The* reduced group C^*-algebra *of a locally compact group G is the closure in operator norm of the image of the group Banach algebra $L^1(G)$ under the (left)-regular representation:*

$$C_r^*(G) = \overline{\pi_{\mathrm{reg}}(L^1(G))} \subset \mathcal{L}(L^2(G)). \tag{1.6}$$

Let D be a G-invariant linear elliptic differential operator on M. If the G-action on M is proper and in addition cocompact one may define an *equivariant analytical index*

$$\mathrm{Ind}_a^G(D) = \text{“}[\mathrm{Ker}\, D] - [\mathrm{CoKer}\, D]\text{”} \in K_0(C_r^*(G)) \tag{1.7}$$

of D. If the kernel and the cokernel of D happen to be finitely generated and projective as modules over $C_r^*(G)$, then the equivariant analytical index of D coincides with their formal difference. As in the classical case the equivariant analytical C^*-index is of topological nature and depends only on the symbol class $[\sigma_{pr}(D)] \in K_G^0(T^*M)$. The G-equivariant topological K-theory for *proper* G-spaces [47] is very similar to the equivariant K-theory with respect to a compact Lie group [3]. In particular, one may define the *equivariant topological index* $\mathrm{Ind}_t^G(D)$ of D as the image of the symbol class under

$$K_G^0(T^*M) \xrightarrow{PD} K_0^G(M) \xrightarrow{\varphi_*} K_0^G(\underline{E}G), \tag{1.8}$$

where PD denotes K-theoretic Poincaré duality and $\varphi : M \to \underline{E}G$ is the equivariant classifying map to a universal proper G-space $\underline{E}G$ [8] (such a space always exists and is unique up to equivariant homotopy equivalence). There is a canonical assembly map [7, 8]

$$\mu : K_*^G(\underline{E}G) \longrightarrow K_*(C_r^*(G)), \tag{1.9}$$

which generalizes (1.5). The corresponding index theorem is

Theorem 1.3 ([8], [31]). — *Let G be a locally compact group and let D be a G-invariant linear elliptic differential operator on the proper, cocompact G-manifold M. Then*

$$\mu(\mathrm{Ind}_t^G(D)) = \mathrm{Ind}_a^G(D). \tag{1.10}$$

Every class in $K_0^G(\underline{E}G)$ can be represented by an equivariant topological index, so that the index theorem characterizes the assembly homomorphism μ as the unique map sending topological to analytical indices.

Baum and Connes conjecture that the assembly map provides the link, which allows a purely geometric description of the K-theory of reduced group C^*-algebras.

Conjecture 1.4 (Baum-Connes Conjecture **(BC)** [8, (3.15)])

Let G be a second countable, locally compact group. Then the assembly map

$$\mu : K_*^G(\underline{E}G) \longrightarrow K_*(C_r^*(G)) \tag{1.11}$$

is an isomorphism of abelian groups.

1.3. The conjecture with coefficients

Baum, Connes, and Higson formulate also a much more general twisted version of conjecture 1.4 [8]. If $D : \mathcal{E}_0 \to \mathcal{E}_1$ is a G-invariant elliptic operator over the proper and cocompact G-manifold M, as considered before, then the topological vector spaces $\mathcal{E}_0$, $\mathcal{E}_1$ are simultaneously modules over G and the C^*-algebra $C_0(M)$ of continuous functions on M vanishing at infinity. One assumes now in addition that

- $\mathcal{E}_0$ and $\mathcal{E}_1$ are (right)-modules over an auxiliary $G - C^*$-algebra A,
- The A-action on $\mathcal{E}_0, \mathcal{E}_1$ commutes with D and the action of $C_0(M)$,
- The module multiplications $C_0(M) \otimes \mathcal{E} \to \mathcal{E}$ and $\mathcal{E} \otimes A \to \mathcal{E}$ are G-equivariant.

These conditions imply that the kernel and the cokernel of D are simultaneously G-modules and A-modules, *i.e.*, they are modules over the following C^*-algebra.

DEFINITION 1.5. — *The* reduced crossed product *of a locally compact group G acting on a C^*-algebra A is the closure (in operator norm) of the image of the twisted group Banach algebra $L^1(G, A)$ under the (left)-regular representation:*

$$C_r^*(G, A) = \overline{\pi_{\mathrm{reg}}(L^1(G, A))} \subset \mathcal{L}(L^2(G, \mathcal{H})), \tag{1.12}$$

$$(f * \xi)(g) = \int_{g'g''=g} \pi(g^{-1} \cdot f(g'))\xi(g'')d\mu, \tag{1.13}$$
$$\forall f \in L^1(G, A), \ \ \forall \xi \in L^2(G, \mathcal{H}),$$

where $\pi : A \to \mathcal{L}(\mathcal{H})$ is any faithful representation. (The algebra $C_r^*(G, A)$ is independent of the choice of π.)

As before, one may define a twisted analytical index

$$\mathrm{Ind}_a^{G,A}(D) \in K_0(C_r^*(G, A)), \tag{1.14}$$

and a twisted topological index

$$\mathrm{Ind}_t^{G,A}(D) \in K_*^G(\underline{E}G, A). \tag{1.15}$$

Here the groups $K_*^G(-, A)$ denote a twisted form of topological K-homology for proper G-spaces. Again, there is a corresponding twisted assembly map, which leads to an index theorem with coefficients.

Example 1.6. — If $G = 1$ and $A = C(X)$, X a compact Hausdorff space, then $K_*(C_r^*(G, A)) \simeq K_*^G(\underline{E}G, A) \simeq K^*(X)$ and the previous index theorem equals the index theorem of Atiyah-Singer [4] for families of elliptic operators parametrized by X.

Baum, Connes, and Higson conjecture that the twisted assembly map allows a geometric description of the K-theory of reduced crossed product C^*-algebras.

CONJECTURE 1.7 (Baum-Connes Conjecture with Coefficients ($\mathbf{BC_{Coeff}}$) [8, (6.9)])

Let G be a second countable locally compact group and let A be a separable G-C^*-algebra. Then the assembly map

$$\mu_{(G,A)} : K_*^G(\underline{E}G; A) \longrightarrow K_*(C_r^*(G, A)) \tag{1.16}$$

from the topological K-homology with coefficients in A of a universal proper G-space $\underline{E}G$ to the K-theory of the reduced crossed product C^*-algebra of (G, A) is an isomorphism of abelian groups.

Remark 1.8. — For $A = \mathbb{C}$ this is just the Baum-Connes conjecture for G.

Remark 1.9. — If $\mathbf{BC_{coeff}}$ holds for a given group G, then it holds for all its closed subgroups H. More specifically, $\mathbf{BC_{coeff}}$ for G and $A = C_0(G/H)$ implies $\mathbf{BC}$ for H.