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PIECEWISE STRONGLY PROXIMAL ACTIONS, FREE BOUNDARIES AND THE NERETIN GROUPS

by Pierre-Emmanuel Caprace, Adrien Le Boudec & Nicolás Matte Bon

Abstract. — A closed subgroup H of a locally compact group G is confined if the closure of the conjugacy class of H in the Chabauty space of G does not contain the trivial subgroup. We establish a dynamical criterion on the action of a totally disconnected, locally compact group G on a compact space X ensuring that no relatively amenable subgroup of G can be confined. This property is equivalent to the fact that the action of G on its Furstenberg boundary is free. Our criterion applies to the Neretin groups. We deduce that each Neretin group has two inequivalent irreducible unitary representations that are weakly equivalent. This implies that the Neretin groups are not of type I, thereby answering a question of Y. Neretin.

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Résumé (*Actions fortement proximales par morceaux, frontières libres et groupes de Neretin*). — Un sous-groupe fermé H d'un groupe localement compact G est confiné si l'adhérence de la classe de conjugaison de H dans l'espace de Chabauty de G ne contient pas le sous-groupe trivial. Nous établissons un critère dynamique sur l'action d'un groupe localement compact totalement discontinu G sur un espace compact X qui garantit que G n'admet pas de sous-groupe relativement moyennable confiné. Cette propriété est équivalente au fait que G agit librement sur sa frontière de Furstenberg. Notre critère s'applique aux groupes de Neretin. Nous déduisons que chaque groupe de Neretin admet des représentations unitaires irréductibles non-équivalentes qui sont faiblement équivalentes. Cela implique que les groupes de Neretin ne sont pas de type I, ce qui répond à une question de Y. Neretin.

1. Introduction

Let G be a locally compact group. A compact G-space X is a compact space equipped with a continuous action of G. The action of G on X is **strongly proximal** if for every $\mu \in \mathrm{Prob}(X)$, the closure of the G-orbit of μ in $\mathrm{Prob}(X)$ contains a Dirac measure, where the space $\mathrm{Prob}(X)$ of Borel probability measures on X is endowed with the weak* topology. The G-space X is a (topological) G**-boundary** if the G-action is minimal and strongly proximal [19]. If G is an amenable group, the only G-boundary is the one-point space, and this property actually characterizes amenability among locally compact groups.

Every group G admits a G-boundary, unique up to isomorphism, with the property that every G-boundary is a factor of it. It is called the **Furstenberg boundary** of G; we denote it by $\partial_F G$. The group G acts faithfully on $\partial_F G$ if and only if the only amenable normal subgroup of G is the trivial subgroup $\langle e \rangle$ [18]. In this note, we are interested in the following:

Problem 1.1. — *Determine when the action of G on $\partial_F G$ is free.*

A key motivation for that question is that the freeness of the G-action on $\partial_F G$ is equivalent to various other properties of the group G. We say that the action of G on a minimal compact G-space X is **topologically free** if there is a dense set of points in X that have a trivial stabilizer in G. We say that H is **relatively amenable** in G if H fixes a probability measure on every compact G-space. Clearly, every amenable subgroup is relatively amenable; when G is discrete, the converse holds; see [11]. A **uniformly recurrent subgroup** (or **URS** for short) of G is a minimal G-invariant closed subset of the Chabauty space $\mathrm{Sub}(G)$ of closed subgroups of G (we recall that $\mathrm{Sub}(G)$ is compact). A closed subgroup $H \leq G$ is **confined** in G if the closure of the conjugacy class of H in $\mathrm{Sub}(G)$ avoids the trivial subgroup $\langle e \rangle$.

Theorem 1.2 (See [7, 25, 27] in the case of discrete groups). — *Let G be a locally compact group. The following conditions are equivalent.*

(i) *The G-action on $\partial_F G$ is free.*
(ii) *There is a G-boundary on which the G-action is topologically free.*
(iii) *No relatively amenable closed subgroup of G is confined.*
(iv) *The only relatively amenable URS of G is the trivial subgroup.*

If in addition G is discrete, then those are also equivalent to:

(v) *G is C^***-simple**, i.e. the reduced C^*-algebra of G is a simple C^*-algebra.*

In the case where the group G is discrete, it follows from the recent works [7, 25, 27] that the five conditions in Theorem 1.2 are equivalent. For a general (e.g. indiscrete) locally compact group G, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are rather straightforward, at least if G is second countable (see Proposition 2.2 and Remark 2.3 below), while the converse implication (i) $\Leftarrow$ (ii) is a particular case of a result from [32]. However, it is an important open problem to determine whether (v) is also equivalent to (i)–(iv). It is worth noting that each of those conditions implies that the group G is totally disconnected (see [39] for (v), and the discussion in the following paragraphs for (iv)).

Conditions (iii) and (iv) highlight an essential feature of Problem 1.1, namely that it can be reformulated in terms of the G-action on the space of its closed subgroups. Indeed, this allows one to avoid constructing explicitly any G-boundary and simply study the conjugation action of G on its relatively amenable subgroups. For example, if G is a discrete hyperbolic group, every amenable subgroup is virtually cyclic, hence finitely generated. In particular, G has countably many amenable subgroups, and it is a general fact that for a group G with this property, the only amenable URS of G is trivial, provided that G has no amenable normal subgroup other than the trivial subgroup. This situation also covers CAT(0) groups by [1, Cor. B]. Many other discrete groups admitting an isometric action satisfying a combination of weak forms of properness and of non-positive curvature can be proved to have a free Furstenberg boundary in a similar (but more elaborate) way; see [23, 14, 7]. Note that the requirement of a certain form of properness cannot be dropped according to [29]. The reformulation of Problem 1.1 in terms of confined subgroups has also been exploited in the realm of discrete groups of dynamical origin in [30].

For non-discrete groups, it turns out that Problem 1.1 has a very different flavour. Indeed, contrary to the discrete case where numerous familiar groups have no non-trivial amenable URS, many natural non-discrete locally compact groups do admit a non-trivial relatively amenable URS. This is notably the case for semi-simple Lie groups and semi-simple algebraic groups over local fields; any such group G indeed has a cocompact amenable subgroup P. By cocompactness, the conjugacy class of P is closed, so P must be confined. Beyond the classical case, in a locally compact group G acting properly and strongly transitively on a locally finite building of arbitrary (not necessarily Euclidean) type, every maximal compact subgroup is confined (this follows from [10, Th. 4.10]). Thus, many natural examples of non-discrete locally compact groups fail to satisfy the condition of Problem 1.1. Note that the above fact for semi-simple Lie groups together with [8, Th. 3.3.3] imply that every locally compact group G, such that the only relatively amenable URS of G is the trivial subgroup, must be totally disconnected.

The first goal of this note is to contribute to Problem 1.1 by establishing a sufficient criterion for a locally compact group G that ensures a positive answer to Problem 1.1. In view of the preceding discussion, there is no loss of generality in restricting it to the case where G is totally disconnected. Given a compact G-space X and a clopen subset α of X, the rigid stabilizer of α in G is the pointwise fixator of $X\backslash\alpha$. It is denoted by $R_G(\alpha) = \mathrm{Fix}_G(X\backslash\alpha)$. We say that the action of G on X is **piecewise minimal-strongly-proximal** if the action of $R_G(\alpha)$ on α is minimal and strongly proximal for every non-empty clopen subset α of X.

Theorem 1.3. — *Let G be a totally disconnected locally compact group. Suppose that there exists a totally disconnected compact G-space X, such that the G-action on X is faithful and piecewise minimal-strongly-proximal. Then G does not have any relatively amenable confined subgroup. Equivalently, G acts freely on its Furstenberg boundary $\partial_F G$.*

Note that the piecewise minimal-strongly-proximal property of the G-action on X implies in particular that X is a G-boundary, and also that this action is very far from being free. Hence, the meaning of Theorem 1.3 is that the existence of a G-boundary that is non-free and satisfies a certain strong compressibility property ensures the existence of another G-boundary where the G-action is free.

Although the reformulation of Problem 1.1 in terms of confined subgroups is helpful, we emphasize that the Chabauty space $\mathrm{Sub}(G)$ and its G-invariant closed subspaces are typically delicate to describe. Moreover, the general properties of the space $\mathrm{Sub}(G)$ are often more subtle in the case of non-discrete groups (see, for instance, [12, §20.1]; see also [17, §1.2] for a remarkable recent result ensuring that, in a simple Lie group of rank ≥ 2, every discrete confined subgroup is a lattice). In the case of discrete groups, Theorem 1.3 is already known [30, Cor. 3.6]. It is actually a consequence of the following more general result: if G is a discrete group, and X a faithful G-space, then for every confined subgroup H of G, there exists a non-empty open subset α of X, such that H contains the commutator subgroup of $R_G(\alpha)$ [31, Th. 1.1]. Here, the assumption made in Theorem 1.3 implies that $R_G(\alpha)$ is non-amenable, so in this situation, it follows in particular that H is not amenable either. However, it is worth noting that, as shown by classical examples, the stronger conclusion of [31, Th. 1.1] completely fails for non-discrete groups; see Section 3.

Examples of groups to which Theorem 1.3 applies are the **Neretin groups** $\mathcal{N}_{d,k}$ of almost automorphisms of quasi-regular tree $\mathcal{T}_{d,k}$. The groups $\mathcal{N}_{d,k}$ are non-discrete, compactly generated, simple, totally disconnected, locally compact groups [9, §6.3]. We refer to [20] for details. The groups $\mathcal{N}_{d,k}$ can be defined as groups of homeomorphisms of the space of ends $\partial\mathcal{T}_{d,k}$, and the action of $\mathcal{N}_{d,k}$ on $\partial\mathcal{T}_{d,k}$ is piecewise minimal-strongly-proximal. The following result is a direct consequence of Theorem 1.3.

Corollary 1.4. — *For all integers $d, k \geq 2$, the Neretin group $\mathcal{N}_{d,k}$ does not have any confined relatively amenable subgroup. In particular, $\mathcal{N}_{d,k}$ does not have any cocompact amenable subgroup.*

To the best of our knowledge, the Neretin group $\mathcal{N}_{d,k}$ is the first known example of a non-discrete, compactly generated, simple locally compact group acting freely on its Furstenberg boundary. It is likely that Theorem 1.3 will apply to many other simple groups.

Using that result, we establish the following representation theoretic property of the Neretin group.

Theorem 1.5. — *For all integers $d, k \geq 2$, the Neretin group $\mathcal{N}_{d,k}$ is not a type I group.*

This answers negatively Question 1.4(2) from [37]. We recall that a locally compact group G is of **type I** is for every unitary representation π, and the von Neumann algebra $\pi(G)''$ is of type I. By Glimm's theorem [22], a second countable group G is of type I if and only if any two weakly equivalent irreducible unitary representations of G are unitarily equivalent. We refer to [15] and [4] for detailed expositions.

Let us also mention that Y. Neretin has proved in [37, Th. 1.2] that the group $\mathcal{N}_{d,d+1}$ has an open subgroup A, such that $(\mathcal{N}_{d,d+1}, A)$ forms a generalized Gelfand pair. Since $\mathcal{N}_{d,d+1}$ admits no cocompact amenable subgroup by Corollary 1.4, we deduce that N. Monod's result on Gelfand pairs [35] cannot be extended to generalized Gelfand pairs, even among simple groups (see Remark 4.3).

Motivated by the understanding of the confined subgroups of $\mathcal{N}_{d,k}$, we also provide a complete classification of the closed cocompact subgroups of $\mathcal{N}_{d,k}$, inspired by [3]. This classification says that there are as few proper cocompact subgroups in $\mathcal{N}_{d,k}$ as one might hope; any such subgroup is a finite index open subgroup of the stabilizer of an end $\xi \in \partial\mathcal{T}_{d,k}$; see Theorem 4.9. This description notably implies that $\mathcal{N}_{d,k}$ is an isolated point of its Chabauty space (see Corollary 4.10). This last phenomenon contrasts with the case of the automorphism group of a regular tree; see Remark 4.11.

The proof of Theorem 1.3 is presented in Section 3. It is fairly elementary. The argument actually establishes non-confinement for an appropriate class