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ON THE FINITENESS OF $\mathfrak{P}$-ADIC CONTINUED FRACTIONS FOR NUMBER FIELDS

by Laura Capuano, Nadir Murru & Lea Terracini

Abstract. — For a prime ideal $\mathfrak{P}$ of the ring of integers of a number field K, we give a general definition of a $\mathfrak{P}$-adic continued fraction, which also includes classical definitions of continued fractions in the field of p-adic numbers. We give some necessary and sufficient conditions on K ensuring that for all but finitely many $\mathfrak{P}$, every $\alpha \in K$ admits a finite $\mathfrak{P}$-adic continued fraction expansion, addressing a similar problem posed by Rosen in the archimedean setting.

Résumé (*Sur la finitude des fractions continues* $\mathfrak{P}$*-adiques pour les corps de nombres*). — Soit $\mathfrak{P}$ un idéal premier de l'anneau des entiers d'un corps de nombres K. On donne une définition générale de fraction continue $\mathfrak{P}$-adique qui inclut les définitions classiques de fractions continues p-adiques. On présente des conditions nécessaires et suffisantes sur K qui assurent que pour tous idéaux $\mathfrak{P}$ sauf un nombre fini, chaque élément $\alpha \in K$ ait une expansion finie en fraction continue $\mathfrak{P}$ -adique. Ces résultats abordent dans le contexte p-adique un problème qui avait été posé par Rosen dans le cas archimédien.

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1. Introduction

The classical continued fraction algorithm provides an integer sequence $[a_0, a_1, \ldots]$ that represents a real number α_0 by means of the following recursive algorithm:

$$\begin{cases} a_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \dfrac{1}{\alpha_n - a_n} \quad \text{if } \alpha_n - a_n \neq 0, \end{cases}$$

for all $n \geq 0$, where $\lfloor \cdot \rfloor$ denotes the integral part of a real number. The a_n's and α_n's are called *partial* and *complete* quotients, respectively. It is easy to see that, for classical continued fractions, the procedure eventually stops if and only if we start with a rational number, and, in the case of irrationals, they provide the best rational approximations of the number; this is one of the reasons why the study of continued fractions is very important in diophantine approximation and transcendence theory.

Motivated by this property, Rosen [34] posed the problem of finding more general definitions of continued fraction expansions characterizing all the elements of an algebraic number field K by means of finite expansions and providing approximations of elements not in the field by means of elements in K (as well as classical continued fractions that provide rational approximations of irrational numbers). In [34], Rosen gave an example of such continued fractions in the special case of $\mathbb{Q}(\sqrt{5})$, using expansions of the form

$$a_0 + \cfrac{b_1}{a_1\varphi + \cfrac{b_2}{a_2\varphi + \cfrac{b_3}{\ddots}}},$$

where φ is the Golden ratio, $b_n = \pm 1$, and the $a_n \in \mathbb{Z}$ satisfies the property that $a_n\varphi$ is the integer multiple of φ nearest to the respective complete quotients. This is a special case of the so-called *Rosen continued fractions*, introduced by the same author in [33], where φ is replaced by irrational numbers of the form $2\cos\frac{\pi}{q}$ with $q \geq 3$ an odd positive number, with the aim of studying Hecke groups.

The characterization of the real numbers having a finite Rosen continued fraction is still an open problem, see, e.g., [1, 9, 18] for further details. In [5], Bernat defined another continued fraction expansion in $\mathbb{Q}(\sqrt{5})$, slightly different from the Rosen one, proving that also these continued fractions represent $\mathbb{Q}(\sqrt{5})$ uniquely. Very recently in [27], the authors generalized a Bernat construction defining the so-called β-continued fraction with the aim of studying when the elements of $\mathbb{Q}(\beta)$ have a finite representation, where β is any quadratic Pisot number. More specifically, the authors proved that, if β is either a quadratic Perron number or the square root of a positive integer, then every element of $\mathbb{Q}(\beta)$ has a finite or eventually periodic β-continued fraction expansion. Moreover, assuming a conjecture by Mercat [29], there exist only four quadratic Perron numbers β, such that the elements of $\mathbb{Q}(\beta)$ have a finite β-continued fraction expansion.

The problem of Rosen can be naturally translated into the context of p-adic numbers $\mathbb{Q}_p$; indeed, starting from Mahler [26], continued fractions have been introduced and studied in $\mathbb{Q}_p$ by several authors. In this context, however, there is no natural definition of a p-adic continued fraction, since there is no canonical definition for a p-adic floor function. The two main definitions of a p-adic continued fraction algorithm are due to Browkin [7] and Ruban [35]; they are both based on the definition of a p-adic floor function

$$s(\alpha) = \sum_{n=k}^{0} x_n p^n \in \mathbb{Q}, \quad \text{where} \;\; \alpha = \sum_{n=k}^{\infty} x_n p^n \in \mathbb{Q}_p,$$

where the x_n's are the representatives modulo p in the interval $(-p/2, p/2)$ for the Browkin definition and in the interval $[0, p-1]$ for the Ruban definition. These continued fractions have been widely studied by several authors in terms of quality of the approximation, finiteness, and periodicity; see, e.g., [2, 4, 8, 10, 11, 39, 21, 32, 36]. In this setting, however, many differences to the classical case arise; for example, none of these definitions provide good approximations as in the real case, and no analogue of Lagrange's theorem holds for both Browkin and Ruban continued fractions, and, hence, the problem of finding a standard definition for a p-adic continued fraction still remains open. However, it has been proved that rational numbers always have a finite Browkin continued fraction expansion [8] and a finite or eventually periodic Ruban continued fraction expansion [21].

In this paper, we consider the p-adic analogue of the Rosen question. Given a number field K and a prime ideal $\mathfrak{P}$ in its ring of integers $\mathcal{O}_K$, we give a very general definition of $\mathfrak{P}$-adic continued fractions; with this definition, the partial quotients are the values of a $\mathfrak{P}$-adic floor function s, which is a locally constant function from the $\mathfrak{P}$-adic completion of K to the ring of $\{\mathfrak{P}\}$-integers of K. We will call the data $\tau = (K, \mathfrak{P}, s)$ a *type* and we will introduce the notion of continued fractions of type τ. With this definition, Browkin and Ruban continued fractions arise as particular p-adic types for $\mathbb{Q}$. If every element of K has a finite (or periodic) τ-expansion, then we shall say that the type τ enjoys the *continued fraction finiteness* (CFF) (or *continued fraction periodicity* (CFP)) property. Moreover, we shall say that the field K enjoys the $\mathfrak{P}$-adic CFF (or CFP) property if there exists a CFF (or CFP) type $\tau = (K, \mathfrak{P}, s)$. It is well known that $\mathbb{Q}$ satisfies the p-adic CFF property for every odd prime p because of the finiteness of Browkin continued fraction expansions of rational numbers (see [7]).

In the first part of the paper, we prove a sufficient condition for a type to have the CFF (or CFP) property using general properties of the multiplicative Weil height of algebraic numbers and of the norms of matrices. This result allows us to study the $\mathfrak{P}$-adic CFF property when K is a norm-Euclidean field; in particular, we prove that a norm-Euclidean field with Euclidean minimum < 1 satisfies the $\mathfrak{P}$-adic CFF property for all but finitely many prime ideals $\mathfrak{P}$. Furthermore, for certain Euclidean quadratic fields K, we provide some more effective constructions by exploiting the form of unitary neighborhoods covering a fundamental domain of $\mathcal{O}_K$ as done in [17].

In the last part of the paper, we study the CFF property of $\mathfrak{P}$-adic continued fractions in relation to the structure of the ideal class group for number fields, which are not necessarily norm-Euclidean. First, we show that, if the number field K satisfies the $\mathfrak{P}$-adic CFF property for all but finitely many prime ideals $\mathfrak{P}$, then $\mathcal{O}_K$ is a PID, giving examples of number fields for which the CFF property fails to hold. Moreover, under milder hypotheses, we show that it is possible to ensure the CFF property for continued fractions associated to (almost all) primes belonging to a norm-Euclidean class, in the sense of [23]. Finally, for a general number field, we show that the obstruction to the CFF property depends on the existence of infinitely many partial quotients with $\mathfrak{P}$-adic valuation equal to -1.

We conclude this Introduction by pointing out some open problems and directions for future work. First, effectiveness: our main results assert the existence of CFF types for a given number field, but in general it is not easy to define them explicitly. The construction of types satisfying the CFF property and the analysis of their properties, such as the study of the arithmetic of partial quotients, or of the dependence between the length of a finite continued fraction and the height of the algebraic number that it represents, are interesting topics that have left as outside of the scope of the present work. Moreover, it would be nice to obtain a full characterization of number fields K satisfying the $\mathfrak{P}$-adic CFF for a given ideal $\mathfrak{P}$. We show that a necessary condition for CFF is that the ideal class group K is generated by the class of $\mathfrak{P}$. We do not know if this condition is also sufficient, but we do not have arguments against this possibility.

Finally, it would be nice to investigate periodicity. Although we state a sufficient condition for a type to enjoy the CFP property, the present paper focuses specifically on finiteness. Nevertheless, periodicity is also a very interesting question, and an algebraic characterization of the elements represented by a periodic expansion (relative to a given type) would be a challenging objective.

2. Notations and prerequisites

For every rational prime p, let $|\cdot|_p$ be the p-adic absolute value, normalized in such a way that $|p|_p = \frac{1}{p}$. The archimedean absolute values on $\mathbb{R}$ or $\mathbb{C}$ will be denoted by $|\cdot|$ or by $|\cdot|_\infty$, respectively. We will denote by $\overline{K}$ an algebraic closure of any field K.

Let K be a number field of degree d over $\mathbb{Q}$ and let $\mathcal{O}_K$ be its ring of integers. We fix a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ lying over an odd prime p. Let $\mathcal{M}_K$ be a set of representatives for the places of K. For every rational prime q and every $v \in \mathcal{M}_K$ above q, let $K_v \subseteq \overline{\mathbb{Q}}_q$ be the completion of K with respect to the v-adic valuation and let $\mathcal{O}_v$ be its valuation ring; we put $d_v = [K_v : \mathbb{Q}_q]$. Let $|\cdot|_v = |N_{K_v/\mathbb{Q}_q}(\cdot)|_q^{\frac{1}{d_v}}$ be the unique extension of $|\cdot|_q$ to K_v. Then the *product formula*

$$\prod_{v \in \mathcal{M}_K} |x|_v^{d_v} = 1$$

holds for all $x \in K^\times$ ([6, Prop. 1.4.4]). We recall the definition of multiplicative Weil height that will be useful in the paper.

DEFINITION 2.1. — For $\alpha \in K$, the (multiplicative) *Weil height* is defined as

$$H(x) = \prod_{v \in \mathcal{M}_K} \sup(1, |x|_v)^{\frac{d_v}{d}}.$$

Notice that all but finitely many factors of the infinite product are equal to 1, and, hence, $H(x)$ is well defined. Moreover, thanks to the choice of the normalization, the definition does not depend on the number field K, and, hence, it extends to a function $H : \overline{\mathbb{Q}} \to [1, +\infty)$. The function H satisfies the following important properties (see [6]):

PROPOSITION 2.2. — *For every nonzero $x, y \in \overline{\mathbb{Q}}$, we have:*

a) $H(x+y) \leq 2H(x)H(y)$;
b) $H(xy) \leq H(x)H(y)$;
c) $H(x^n) = H(x)^{|n|}$ *for all* $n \in \mathbb{Z}$;
d) $H(\sigma(x)) = H(x)$ *for all* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
e) **Northcott's theorem:** *there are only finitely many algebraic numbers of bounded degree and bounded height;*
f) **Kronecker's theorem:** $H(x) = 1$ *if and only if x is a root of unity.*

3. $\mathfrak{P}$-adic continued fractions

In this section, we show, given a number field K and a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$, how to define a general $\mathfrak{P}$-continued fraction. Our general definition will generalize the classical definitions of p-adic continued fractions given by Browkin and Ruban.