Séminaire BOURBAKI Janvier 2020
72e année, 2019–2021, no 1170, p. 131 à 157
doi : 10.24033/ast.1159

HODGE THEORY AND O-MINIMALITY
[after B. Bakker, Y. Brunebarbe, B. Klingler, and J. Tsimerman]

by **Javier Fresán**

INTRODUCTION

Envisioned by Grothendieck (1984) as a way out of the pathologies that one encounters when dealing with all topological spaces, *tame topology* has reached maturity over the last decades through the study of o-minimal structures in model theory. In a nutshell, attention is restricted to those topological spaces obtained by gluing finitely many subsets of $\mathbf{R}^n$ that are definable by first order formulas involving the operations and the order coming from the real numbers, as well as functions of a certain class chosen beforehand. The collection of such sets is called a structure, and one says that a structure is *o-minimal* if the only definable subsets of $\mathbf{R}$ are finite unions of points and open intervals. For example, the structure $\mathbf{R}_{\mathrm{an,exp}}$ in which the class of functions consists of real analytic functions on the unit hypercube and the real exponential is o-minimal. In developing a complex geometry with definable open subsets as charts, this axiom allows for global algebraicity results without renouncing the local flexibility of analytic varieties, as is best illustrated by the o-minimal Chow theorem of Peterzil and Starchenko (2009): if a closed analytic subset of a complex algebraic variety is definable in some o-minimal structure, then it is automatically algebraic, whether the ambient space is proper or not. In a slightly different direction, a celebrated theorem of Pila and Wilkie (2006) affirms that definable subsets of $\mathbf{C}^n$ with many rational points of bounded height necessarily contain non-trivial semialgebraic subsets on which most of these points will lie. By means of this result, o-minimality has found spectacular applications to diophantine geometry. My aim in this survey is to convey the idea that it has very recently become an important tool to understand Hodge theory as well.

Our main object of interest will be the *period maps* describing how Hodge structures vary on a family of smooth projective varieties. As a case study, one may think of the Legendre family of elliptic curves parameterised by the punctured projective line $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The fibres $\mathcal{E}_s$ are the projective completions of the affine plane curves $y^2 = x(x-1)(x-s)$. On a small neighbourhood around each point of S, all fibres are canonically diffeomorphic, so we may choose a common symplectic basis γ_1, γ_2 of the first homology group $H_1(\mathcal{E}_s, \mathbf{Z})$. By contrast, the position of the line $\mathbf{C}\omega \subset H^1(\mathcal{E}_s, \mathbf{C})$ spanned by the holomorphic differential $\omega = \mathrm{d}x/y$ will vary as s moves, for it encodes the complex structure on $\mathcal{E}_s$. This is our first example of a polarised variation of pure Hodge structures. Concretely, the line in question is determined by the ratio $\int_{\gamma_2} \omega / \int_{\gamma_1} \omega$ of the two periods of the form ω, and this gives a multivalued map from S to the upper half-plane $\mathfrak{H}$. The monodromy being governed by the congruence subgroup $\Gamma(2)$ of $\mathbf{SL}_2(\mathbf{Z})$, it descends to a holomorphic map from S to the modular curve $\Gamma(2)\backslash\mathfrak{H}$. In this very special case, the target is an algebraic variety and the period map is even an isomorphism.

For more general families, the role of $\mathfrak{H}$ is played by a homogeneous space G/M classifying polarised Hodge structures of the same type as those on the cohomology of the fibres, the modular curve is replaced by the quotient $S_{\Gamma,G,M}$ of G/M by an arithmetic subgroup $\Gamma \subset G$, and the period map is a holomorphic map from the parameter space to $S_{\Gamma,G,M}$. As soon as one leaves the realm of abelian varieties, these arithmetic quotients are complex analytic spaces which almost never carry an algebraic structure, so the holomorphic, non-algebraic period maps could a priori behave wildly at infinity. Nevertheless, Bakker, Klingler, and Tsimerman (2020) show that all period maps have tame geometry: they are definable in the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$ relatively to a natural semialgebraic structure on $S_{\Gamma,G,M}$. From this and the o-minimal Chow theorem, they derive a new proof of the algebraicity of Hodge loci, originally a theorem by Cattani, Deligne, and Kaplan (1995). As another striking application of definability of period maps, along with a new o-minimal GAGA theorem, Bakker, Brunebarbe, and Tsimerman (2018) recently established a long-standing conjecture of Griffiths to the effect that their images are quasi-projective algebraic varieties. Things are rapidly moving and I feel other breakthroughs are to come.

The text is organised as follows. In section 1, we recall the construction of the period map associated with a polarised variation of pure Hodge structures. Section 2 starts with a very brief introduction to o-minimal structures, before turning to the o-minimal Chow and the o-minimal GAGA theorems. After introducing the key notion of Siegel sets, we prove that arithmetic quotients admit a functorial semialgebraic definable structure in section 3. Then section 4 is devoted to the proof of definability of period maps, which relies on a fine description of their asymptotic behaviour near the boundary. Finally, we present the applications to algebraicity of Hodge loci and Griffiths's conjecture in sections 5 and 6 respectively. I recommend the lecture notes by Bakker (2019) as a complementary reading.

Acknowledgments. — I would like to thank Benjamin Bakker, Olivier Benoist, Yohan Brunebarbe, Bruno Klingler, Emmanuel Kowalski, and Marco Maculan for their help in preparing these notes, as well as all the speakers and participants of the reading seminar "Théorie de Hodge et o-minimalité" at the Université d'Orsay and the summer school "o-minimal structures in algebraic geometry" at the University of Freiburg, where I first learnt part of the material.

1. VARIATIONS OF HODGE STRUCTURES AND PERIOD MAPS

1.1. Polarised pure Hodge structures

Let k be an integer. A *pure Hodge structure H of weight k* is a finitely generated abelian group $H_{\mathbf{Z}}$ together with a bigrading

$$H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p+q=k} H^{p,q}$$

satisfying $\overline{H^{p,q}} = H^{q,p}$, where barring stands for complex conjugation. On setting

$$F^p = \bigoplus_{r \geq p} H^{r,s},$$

these data amount to a finite decreasing filtration $F^\bullet$ of $H_{\mathbf{C}}$ such that $F^p \oplus \overline{F^{k+1-p}} = H_{\mathbf{C}}$ holds for all p. The dimensions $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}$ are called the *Hodge numbers*, and $F^\bullet$ is called the *Hodge filtration*. Yet another equivalent way of thinking of Hodge structures is as representations

$$\varphi \colon \mathbf{S} \longrightarrow \mathbf{GL}(H_{\mathbf{R}})$$

of Deligne's torus $\mathbf{S}$, which is the real algebraic group of invertible matrices of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Being pure of weight k is then the condition that the diagonal subtorus $\left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$ acts through the homothety of ratio t^k, and $H^{p,q}$ is recovered as the eigenspace for the character $z \mapsto z^{-p} \cdot \overline{z}^{-q}$ of $\mathbf{S}(\mathbf{R}) \cong \mathbf{C}^\times$ acting on $H_{\mathbf{C}}$. A morphism of Hodge structures is a homomorphism of the underlying abelian groups that is compatible with the Hodge filtration, or equivalently with the action of $\mathbf{S}$.

Let $q_{\mathbf{Z}} \colon H_{\mathbf{Z}} \times H_{\mathbf{Z}} \to \mathbf{Z}$ be a bilinear form which is symmetric if k is even and alternating if k is odd. The associated *Hodge form* is the hermitian form

$$h \colon H_{\mathbf{C}} \times H_{\mathbf{C}} \longrightarrow \mathbf{C}, \quad h(u,v) = q_{\mathbf{C}}(Cu, \overline{v}),$$

where C is the Weil operator acting as multiplication by i^{p-q} on the summand $H^{p,q}$. We say that $q_{\mathbf{Z}}$ is a *polarisation* on H, or that H is polarised by $q_{\mathbf{Z}}$, if the Hodge decomposition is h-orthogonal and h is positive-definite; in other words, if

a) $h(u,v) = 0$ whenever u and v lie in different subspaces $H^{p,q}$,

b) $h(u,u) > 0$ for all non-zero $u \in H_{\mathbf{C}}$.

In particular, $q_{\mathbf{Q}}$ is non-degenerate. The above conditions, which generalise the classical Riemann relations for abelian varieties, are often referred to as *bilinear Hodge–Riemann relations*. In terms of the Hodge filtration, a) says that the orthogonal complement of F^p with respect to $q_{\mathbf{Z}}$ is precisely F^{k+1-p}. If $\mathbf{Z}(-k)$ denotes the Hodge structure of weight $2k$ on $\mathbf{Z}$, it also amounts to asking that

$$q_{\mathbf{Z}} \colon H \otimes H \longrightarrow \mathbf{Z}(-k)$$

is a morphism of Hodge structures. When a polarisation exists, we say that H is *polarisable*.

Example 1.1. Let X be a smooth projective complex variety of dimension n. Singular cohomology $H^k(X, \mathbf{Z})$ carries a polarisable pure Hodge structure of weight k. Upon identifying its complexification $H^k(X, \mathbf{C})$ with algebraic de Rham cohomology $H^k(X, \Omega_X^\bullet)$, the Hodge filtration is given by

$$F^p = \mathrm{Im}\big(H^k(X, \Omega_X^{\bullet \geq p}) \longrightarrow H^k(X, \mathbf{C})\big).$$

Polarisations come from choosing the class of a hyperplane section $\eta \in H^2(X, \mathbf{Z})$ and considering the Lefschetz operator $L \colon H^*(X, \mathbf{Z}) \to H^{*+2}(X, \mathbf{Z})$ given by cup product with η. For each $j \leq n$, the j-th *primitive* cohomology is defined as

$$P^j(X, \mathbf{Z}) = \ker\big(L^{n-j+1} \colon H^j(X, \mathbf{Z}) \to H^{2n-j+2}(X, \mathbf{Z})\big),$$

which is a sub-Hodge structure of $H^j(X, \mathbf{Z})$. According to the Lefschetz theorems, it is polarised by the intersection form

$$q_{\mathbf{Z}}^j \colon P^j(X, \mathbf{Z}) \times P^j(X, \mathbf{Z}) \longrightarrow \mathbf{Z}, \quad (\alpha, \beta) \longmapsto (-1)^{\frac{1}{2}j(j-1)} \int_X \eta^{n-j} \cdot \alpha \cdot \beta,$$

and the whole cohomology in each degree k decomposes rationally as the direct sum

$$H^k(X, \mathbf{Q}) = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} L^i P^{k-2i}(X, \mathbf{Q}),$$

where the Lefschetz operator and primitive cohomology are now taken with rational coefficients and $P^j(X, \mathbf{Q}) = 0$ for all $j > n$. Modifying the signs as $(-1)^i q_{\mathbf{Q}}^{k-2i}$ on the i-th summand gives rise to a polarisation on $H^k(X, \mathbf{Q})$ that, after clearing denominators by multiplying by a sufficiently large integer, induces a polarisation on $H^k(X, \mathbf{Z})$.

1.2. Period domains

The book by Carlson, Müller-Stach, and Peters (2017) is an accessible reference for this section. Fix an integer k, a finitely generated abelian group $H_{\mathbf{Z}}$ of rank r, a bilinear form $q_{\mathbf{Z}}$ on $H_{\mathbf{Z}}$ which is symmetric if k is even and alternating if k is odd, and a collection of non-negative integers $\{h^{p,q}\}_{p+q=k}$ such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = r$. Associated with these data is a *period domain* $\mathcal{D}$ classifying pure Hodge structures of weight k on $H_{\mathbf{Z}}$ which are polarised by $q_{\mathbf{Z}}$ and have Hodge numbers $h^{p,q}$.

Although $\mathcal{D}$ is a priori only a set, it can be endowed with the structure of a complex manifold as follows. Setting $f^p = \sum_{r \geq p} h^{p,q}$, one first considers the *compact dual*

$$\check{\mathcal{D}} = \left\{ \begin{array}{c} \text{finite decreasing filtrations } F^\bullet \text{ on } H_{\mathbf{C}} \text{ such that} \\ (F^p)^\perp = F^{k+1-p} \text{ and } \dim F^p = f^p \end{array} \right\}, \tag{1}$$

which is a closed analytic subset of the product of Grassmannians $\prod_p \mathrm{Gr}(f^p, H_{\mathbf{C}})$, and hence a projective complex variety. The period domain is the open subset $\mathcal{D} \subset \check{\mathcal{D}}$ consisting of those filtrations for which the Hodge form is positive-definite.

Let $\mathbf{G} = \mathrm{Aut}(H_{\mathbf{Q}}, q_{\mathbf{Q}})$ be the group of automorphisms $g \in \mathbf{GL}(H_{\mathbf{Q}})$ which are compatible with the polarisation in that $q_{\mathbf{Q}}(g(x), g(y)) = q_{\mathbf{Q}}(x, y)$ for all $x, y \in H_{\mathbf{Q}}$. It is a semisimple linear algebraic group over $\mathbf{Q}$. By an elementary argument in linear algebra, its complex points $\mathbf{G}(\mathbf{C})$ operate transitively on $\check{\mathcal{D}}$. The compact dual is hence smooth and the period domain inherits the structure of a complex manifold from it. More is true: the subgroup $\mathbf{G}(\mathbf{R})$ preserves $\mathcal{D} \subset \check{\mathcal{D}}$ and the induced action is transitive as well. If we choose some base point of $\mathcal{D}$ and we let B and M denote its stabilisers in $\mathbf{G}(\mathbf{C})$ and $G(\mathbf{R})$ respectively, the period domain can be realised as the quotient

$$\mathcal{D} = \mathbf{G}(\mathbf{R})/M \hookrightarrow \check{\mathcal{D}} = \mathbf{G}(\mathbf{C})/B.$$

Since M consists of real elements and $H^{p,q} = F^p \cap \overline{F^q}$, it not only leaves the Hodge filtration invariant but also the Weil operator and thus the Hodge form; as any isotropy group of a positive-definite hermitian form, M is hence a compact subgroup of $G(\mathbf{R})$.

Example 1.2. If $k = 1$ and the only non-zero Hodge numbers are $h^{1,0} = h^{0,1} = g$, the period domain is the subset of $\mathrm{Gr}(g, H_{\mathbf{C}})$ consisting of totally isotropic subspaces F^1 on which the hermitian form $\mathrm{i}\, q_{\mathbf{C}}(u, \overline{u})$ is positive-definite. After choosing a symplectic basis $\{e_1, \ldots, e_g, f_1, \ldots, f_g\}$ of $H_{\mathbf{C}}$, each F^1 has a unique basis of the form

$$\omega_i = e_i + \sum_{j=1}^{g} z_{j,i} f_j \qquad (i = 1, \ldots, g),$$

and it follows from the bilinear Hodge–Riemann relations that the complex $g \times g$ matrix $Z = (z_{i,j})$ is symmetric and has positive-definite imaginary part. Therefore, the period domain $\mathcal{D}$ is in bijection with Siegel's upper half-space

$$\mathfrak{H}_g = \big\{ g \times g \text{ symmetric matrices } Z = X + iY \text{ with } Y \text{ positive-definite} \big\}.$$

In this case, $\mathbf{G} = \mathbf{Sp}_{2g}$ is the symplectic group, $M = \mathbf{U}_g$ is a maximal compact subgroup of its real points, and $\mathfrak{H}_g = \mathbf{G}(\mathbf{R})/M$ is a hermitian symmetric domain.

1.3. Variations of polarised pure Hodge structures

Let S be a smooth connected quasi-projective complex variety. By a *local system* on S we mean a locally constant sheaf $\mathcal{V}_{\mathbf{Z}}$ of finitely generated abelian groups on $S(\mathbf{C})$. Upon choosing a base point $s_0 \in S$, giving a local system on S amounts to giving a representation

$$\rho \colon \pi_1(S, s_0) \longrightarrow \mathbf{GL}(\mathcal{V}_{\mathbf{Z}, s_0})$$

of the fundamental group based at s_0, which is called the *monodromy representation*. Another incarnation of the local system $\mathcal{V}_{\mathbf{Z}}$ is the holomorphic flat vector bundle

$$(\mathcal{V}_{\mathcal{O}}, \nabla) = (\mathcal{V}_{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}_S} \mathcal{O}_S, \mathrm{id} \otimes d).$$

An example to keep in mind arises from families of algebraic varieties parameterised by S. Namely, if $f \colon \mathcal{X} \to S$ is a smooth projective morphism from a smooth quasi-projective variety $\mathcal{X}$, the sheaf

$$\mathcal{V}_{\mathbf{Z}} = R^k f_* \underline{\mathbf{Z}}_{\mathcal{X}}$$