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JOININGS CLASSIFICATION AND APPLICATIONS
[*after* Einsiedler and Lindenstrauss]

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Introduction

Fix a group A and consider a set of measure-preserving actions of A on $\mathrm{X_i} = (X_i, \mathscr{B}_i, \mu_i)$, $i = 1, \dots, r$, where X_i is a Borel probability space with a measure μ_i and a σ-algebra $\mathscr{B}_i$. Consider the joint action (also called the diagonal action) of A on

$$\mathrm{X} = (X_1 \times \cdots \times X_r, \mathscr{B}_1 \times \cdots \times \mathscr{B}_r)$$

given by $a.(x_1, \dots, x_r) = (a.x_1, \dots, a.x_r)$. A ($r$-fold) *joining* of the systems $\{\mathrm{X_i}\}_{i=1}^r$ is an A-invariant probability measure μ on X with $(\pi_i)_* \mu = \mu_i$ for $i = 1, \dots, r$ where $\pi_i \colon X \to X_i$ is the natural projection map.

There always exists at least one joining, namely the *trivial joining*, which is the product measure $\mu_1 \otimes \cdots \otimes \mu_r$. When this is the only possible joining of the systems $\{\mathrm{X_i}\}_{i=1}^r$ one says that these systems are *disjoint*. The systematic study of joinings stems from Furstenberg's seminal paper (Furstenberg, 1967). Furstenberg marked an analogy between joinings and the arithmetic of integers: saying that two measure-preserving systems are disjoint is analogous to saying that their least common multiple is their product. The analogy works in one direction; measure-preserving systems admitting a non-trivial common factor are never disjoint: recall that a *factor* of a measure-preserving system $\mathrm{X} = (X, \mathscr{B}, \mu, A)$ is a measure-preserving system $\mathrm{Y} = (Y, \mathscr{C}, \nu, A)$ and a measure-preserving map $\phi \colon X \to Y$ which intertwines the action of A, that is, for all $a \in A$ we have $a.\phi(x) = \phi(a.x)$ for μ-almost everywhere. Like integers, any measure-preserving system has itself and the trivial system (one-point system) as factors. Moreover, like integers, as stated above, if two measure-preserving systems have a common factor, they have a non-trivial joining, called the relatively independent joining over a common factor (see e.g., Einsiedler and Ward, 2011, §6.5). Furstenberg (1967) asked if this analogy also works in the other direction: if two systems do not have any common factor, must they be disjoint? Rudolph (1979) answered negatively, providing the first counterexamples. Joinings are nonetheless a strong tool in ergodic theory, as exemplified by Glasner (2003) which gives a complete treatment of ergodic theory via joinings. The broad applicability of the classification of possible joinings of certain systems was already visible in the work of Furstenberg (1967), where he solves a question in Diophantine approximation using joinings. We refer the reader also to the recent survey of de la Rue (2020) about the broad use of joinings in ergodic theory.

Roughly said, the study of joinings is the study of all possible ways two systems (or r systems) can be embedded as factors of another system, which is in turn spanned by them. When two systems are *not* disjoint, this is a sign that there is strong relation between them. The main topic of this survey is a very good example of this principle. This is a survey of the work of Einsiedler and Lindenstrauss on joinings of higher-rank torus actions on S-arithmetic homogeneous spaces (Einsiedler and Lindenstrauss, 2019), which extends their previous paper (Einsiedler and Lindenstrauss, 2007). They consider torus actions on two (or r) homogeneous spaces which are quotients of S-arithmetic points of perfect algebraic groups, equipped with the uniform Haar probability measure on each quotient. They show in particular that if such systems are not disjoint, there must be a strong *algebraic* relation between the corresponding perfect algebraic groups, exemplifying the principle stated above. This may remind the reader of the folklore Goursat's Lemma from group theory; while the latter is a natural structural theorem about subgroups of a product, the joining theorem of Einsiedler and Lindenstrauss is a striking instance of measure rigidity, where the existence of non-trivial joinings in this setting can only be due to a strong algebraic relation.

The main result of Einsiedler and Lindenstrauss (2019, Theorem 1.7) classifies joinings on higher-rank torus actions on a product of two (or r) homogenous spaces of the form

$$\Gamma_1\backslash\mathbb{G}_1(\mathbb{Q}_S)\times\Gamma_2\backslash\mathbb{G}_2(\mathbb{Q}_S)$$

as we now state after recalling the necessary definitions. The measure spaces we consider are S-arithmetic homogeneous quotients of perfect groups. More precisely, let $\mathbb{G}$ be a perfect Zariski-connected linear algebraic group defined over $\mathbb{Q}$ and let S be a finite set of places of $\mathbb{Q}$. Let $\mathbb{Q}_S$ denote $\prod_{s\in S}\mathbb{Q}_s$ (with $\mathbb{Q}_\infty=\mathbb{R}$). An S-arithmetic quotient is a quotient space of the form $\Gamma\backslash G$ with G being a finite-index subgroup of $\mathbb{G}(\mathbb{Q}_S)$ and Γ is an irreducible arithmetic lattice commensurable to $\mathbb{G}(\mathscr{O}_S)$. Here, $\mathscr{O}_S$ denotes the ring of S-adic integers. Such an S-arithmetic quotient is said to be *saturated by unipotent* if the group generated by all unipotent elements of G acts ergodically on $\Gamma\backslash G$. For example, for $\mathbb{G}=\mathrm{SL}_n$ (or more generally simply-connected algebraic groups) the quotient $\Gamma\backslash\mathbb{G}(\mathbb{Q}_S)$ is saturated by unipotents.

A probability measure μ on an S-arithmetic quotient $\Gamma\backslash G$ is called *algebraic over* $\mathbb{Q}$ if there exists an algebraic group $\mathbb{H}$ defined over $\mathbb{Q}$ and a finite-index subgroup $H<\mathbb{H}(\mathbb{Q}_S)$ such that $\mu=m_{\Gamma Hg}$ where $g\in G$ and $m_{\Gamma Hg}$ denotes the normalized Haar measure on a single (necessarily closed, by the finiteness of μ) orbit - see §2.1 for a detailed definition.

The joinings we aim to classify are joinings of S-arithmetic quotients $X_i = \Gamma_i \backslash G_i$ which are saturated by unipotents, equipped with Haar probability measures $m_{X_i} = m_{\Gamma\backslash G}$, and a torus action which we now define. Following the notation of Einsiedler and Lindenstrauss (2019) we say that a subgroup $A < G$ is of class-$\mathscr{A}'$ if it is simultaneously diagonalizable and the projection of $a \in A$ to $\mathbb{G}(\mathbb{Q}_s)$ for any $s \in S$ satisfies the following: for $s = \infty$ it has only positive real eigenvalues, and for s equal to a finite prime p, we assume that all the eigenvalues are powers of θ_p for some $\theta_p \in \mathbb{Q}_p^\times$ with $\left|\theta_p\right|_p \neq 1$ chosen independently of $a \in A$. A homomorphism $\phi\colon \mathbb{Z}^d \to G$ is said to be of class $\mathscr{A}'$ if it is proper and $\phi(\mathbb{Z}^d)$ is of class-$\mathscr{A}'$. The term *higher-rank torus action* refers to such a homomorphism with $d \geqslant 2$. We are ready to state the main theorem of Einsiedler and Lindenstrauss (2019, Theorem 1.7):

Theorem 1.1 (Einsiedler–Lindenstrauss, 2019). *Let $r, d \geqslant 2$ and let $\mathbb{G}_1, \ldots, \mathbb{G}_r$ be perfect algebraic groups defined over $\mathbb{Q}$, $\mathbb{G} = \prod \mathbb{G}_i$, and S be a finite set of places of $\mathbb{Q}$. Let $X_i = \Gamma_i \backslash G_i$ be S-arithmetic quotients for $G_i < \mathbb{G}_i(\mathbb{Q}_S)$ which are saturated by unipotents and set $G = \prod_{i=1}^r G_i$ and $X = \prod_{i=1}^r X_i$. Let $\phi_i\colon \mathbb{Z}^d \to G_i$ be homomorphisms such that $\phi = (\phi_1, \ldots, \phi_r)\colon \mathbb{Z}^d \to G$ is of class-$\mathscr{A}'$, and such that the projection of ϕ_i to every $\mathbb{Q}$-almost simple factor of $\mathbb{G}_i(\mathbb{Q}_S)$ is proper. Let $A = \phi(\mathbb{Z}^d)$ and suppose μ is an A-invariant and ergodic joining of the actions of $A_i = \phi_i(\mathbb{Z}^d)$ on X_i equipped with the Haar measure m_{X_i}. Then, μ is an algebraic measure defined over $\mathbb{Q}$.*

This theorem exemplifies the above principle concerning disjointness: let $\mathbb{H} < \mathbb{G}$ be the group showing the algebraicity of μ. If $\mathbb{H} = \mathbb{G}$ then μ is the trivial joining. Otherwise, $\mathbb{H}$ arises from a very strong relation between the algebraic groups $\mathbb{G}_i$. Indeed, certain of their $\mathbb{Q}$-simple factors need to be isogenous *over* $\mathbb{Q}$. In particular, if $\mathbb{G}_i$ are pairwise non-$\mathbb{Q}$-isogenous almost simple groups, any joining must be the trivial one. This situation strongly echoes Goursat's Lemma from group theory.

Taking again the broader viewpoint of measure rigidity for torus action (or $\mathbb{Z}^d$-actions) on homogeneous spaces, Theorem 1.1 is the most complete result in this context. Such rigidity results are currently only possible under a positive entropy assumption. In our context, the positive entropy assumption is hidden in the assumption that we join homogeneous spaces equipped with the Haar probably measure on each quotient (we give more details below). Moreover, the assumption that the groups are perfect is essential: considering more general groups in both factors would allow to recast the classification of $\mathbb{Z}^k$-actions on solenoids (including the zero entropy case - a notoriously difficult problem), as a classification problem of joinings.

Theorem 1.1 is already interesting when $r = 2$ and $\mathbb{G}_1 = \mathbb{G}_2 = \mathrm{SL}_n$ for $n \geqslant 3$ and $d \geqslant 2$, or for $\mathbb{G}_1 = \mathbb{G}_2 = \mathrm{SL}_2 \times \mathrm{SL}_2$ and $d = 2$. While reading this survey, the reader is advised to concentrate on these cases. Indeed, the techniques used and the main steps of the proofs of Einsiedler and Lindenstrauss (2007) and Einsiedler and Lindenstrauss (2019) are already visible when one considers the case where $\mathbb{G}_1$

and $\mathbb{G}_2$ are equal to SL_n for $n \geqslant 3$ or to $\mathrm{SL}_2 \times \mathrm{SL}_2$, and where $S = \{\infty\}$, that is, where we consider real Lie groups. Therefore, apart from describing the main result of Einsiedler and Lindenstrauss (2019) in this introduction, we will reduce this survey to these cases.

To end this introduction we present a few images of the following arithmetic application (Aka, Einsiedler, and Shapira, 2016) which appeared at the same time as (Einsiedler and Lindenstrauss, 2019). We discuss further applications in §6.

For $D \in \mathbb{N}$ write

$$\mathbb{S}^2(D) = \left\{(x,y,z) \in \mathbb{Z}^3 : x^2+y^2+z^2 = D, \gcd(x,y,z) = 1\right\}.$$

By Legendre and Gauss we have $\mathbb{S}^2(D) \neq \emptyset$ if and only if $D \neq 0,4,7 \mod 8$. Consider

$$P_D := \frac{1}{\sqrt{D}} \cdot \mathbb{S}^2(D) \subset \mathbb{S}^2 := \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\}. \tag{1}$$

By a celebrated theorem of Duke (1988), based on a breakthrough of Iwaniec (1987), P_D equidistribute on $\mathbb{S}^2$ when $D \to \infty$ along $D \neq 0,4,7 \mod 8$. That is, the following weak-* convergence

$$\mu_D := \frac{1}{|\mathbb{S}^2(D)|} \sum_{v \in \mathbb{S}^2(D)} \delta_{\frac{v}{\sqrt{D}}} \longrightarrow m_{\mathbb{S}^2}$$

holds, where $m_{\mathbb{S}^2}$ is the uniform (cone) measure on $\mathbb{S}^2$.

We wish to join this equidistribution problem with another equidistribution problem in a natural way. For each $v \in \mathbb{S}^2(D)$ we consider the two-dimensional lattice $\Lambda_v := v^\perp \cap \mathbb{Z}^3$ which we can consider up to rotation as lying in a fixed plane of $\mathbb{Q}^3$. We denote it by $[\Lambda_v]$ and call it *the (shape of the) orthogonal lattice of v*. The set $Q_D := \{[\Lambda_v] : v \in \mathbb{S}^2(D)\}$ can be considered as a subset of the modular surface $X_2 := \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ which parametrizes the space of two-dimensional lattices up to rotation, and carries a natural invariant probability measure m_{X_2}. A careful analysis (see e.g., Ellenberg, Michel, and Venkatesh, 2013, §5.2) shows that the normalized counting measure on Q_D also equidistributes as $D \to \infty$ to m_{X_2}, by a variant of Duke's Theorem. This construction yields the following natural problem: does the normalized counting measure on

$$J_D := \left\{(v, [\Lambda_v]) : v \in \mathbb{S}^2(D)\right\} \tag{2}$$

equidistribute to the product measure $m_{\mathbb{S}^2} \otimes m_{X_2}$ when $D \to \infty$ with $D \neq 0,4,7 \mod 8$?

We conjecture that it does (mainly because we don't see a reason why it shouldn't). Here is some "visible" evidence: for the D's below, we divide the modular surface X_2 using the height function into two (resp. three) equal m_{X_2}-measure regions and call lattices in each region non-stretched/stretched (resp. non-stretched/mildly stretched/super-stretched) and color each point on $\frac{1}{\sqrt{D}} \cdot \mathbb{S}^2(D)$ with a different color according to the type of its orthogonal lattice. In figures 1 and 2 below, one can see the distribution of the corresponding points together with the number of points of each type for $D = 101, 8011, 104851, 14500001$.

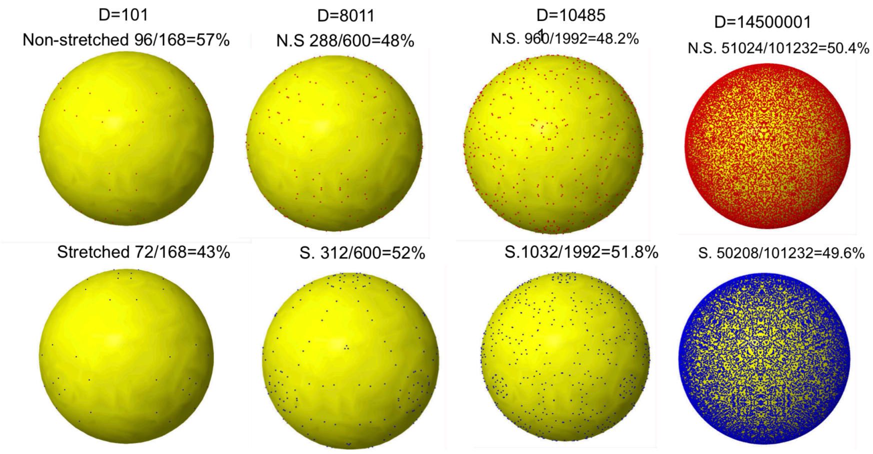

Figure 1: non-stretched vs. stretched

Both equidistribution problems in $\mathbb{S}^2$ and in X_2 may be individually phrased as two individual equidistribution problems on an S-arithmetic (or adelic) quotient as defined above (see § 6.1 for more details). Linnik could prove these results under a congruence condition on D modulo a fixed arbitrary prime (see § 6.1 for more details). It turns out that the coupling of $v \in \mathbb{S}^2(D)$ with its orthogonal lattice $[\Lambda_D]$ gives rise to a joining of the above S-arithmetic quotient. Under congruence conditions at *two* fixed arbitrary primes, one could apply Theorem 1.1 to deduce the equidistribution of the normalized counting measure on J_D to $m_{\mathbb{S}^2} \otimes X_2$ when $D \to \infty$ along $D \neq 0, 4, 7 \mod 8$ and the congruence conditions modulo the above two fixed primes. Recently Blomer and Brumley (2020) showed that under the Generalized Riemann Hypothesis, the above equidistribution holds along $D \neq 0, 4, 7 \mod 8$ without any congruence conditions.

1.1. Bird's-eye view and the organization of this survey

Let's give a (very subjective and entropy-centred) bird's-eye view of the main ingredients used in the proofs of the main theorems of the works Einsiedler and Lindenstrauss (2007, 2019). There are four main ingredients, all related to entropy: