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LOCAL MARKED LENGTH SPECTRUM RIGIDITY
[*after* Guillarmou and Lefeuvre]

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1. Introduction

The search for characterizing a smooth Riemannian metric on a smooth closed manifold M by easy to define geometric quantities has a long and fruitful history, usually described as rigidity problems.

A particularly appealing rigidity problem can be formulated as follows. Consider a closed manifold M of dimension $n \geqslant 2$, equipped with a Riemannian metric g_0 of nonpositive sectional curvature. By the Hadamard Cartan theorem, the universal covering $\tilde{M}$ of M is diffeomorphic to $\mathbb{R}^n$ and hence M is a classifying space for its fundamental group $\pi_1(M)$. Each nontrivial conjugacy class c in $\pi_1(M)$ can be represented by a closed geodesic γ_c, of minimal length $L(\gamma_c)$ in the corresponding free homotopy class. If we denote by $\mathscr{C}$ the set of all conjugacy classes in $\pi_1(M)$, then the metric g_0 determines a function $L_{g_0} : \mathscr{C} \to (0, \infty)$ by defining $L_{g_0}(c) = L_{g_0}(\gamma_c)$ $(c \in \mathscr{C})$. This function is called the *marked length spectrum* of g_0. It also makes sense for metrics on M which are not nonpositively curved.

The following conjecture was formulated by Burns and Katok Burns and Katok (1985) but may have been known earlier.

Conjecture 1.1. *Let g_0 be a negatively curved Riemannian metric on a closed manifold M. If g is another metric on M so that $L_g = L_{g_0} : \mathscr{C} \to (0, \infty)$, then g, g_0 are strongly isometric.*

Here two metrics g, g_0 are called *strongly isometric* if there exists a diffeomorphism ϕ isotopic to the identity such that $\phi^* g = g_0$. The following major progress towards this conjecture is the main result of Guillarmou and Lefeuvre (2019).

Theorem 1.2 (Guillarmou and Lefeuvre). *Let g_0 be a smooth nonpositively curved metric on a closed manifold M of dimension n whose geodesic flow is Anosov. Then there exists a neighborhood U of g_0 in the C^N-topology for some $N > 3n/2 + 8$ such that any metric in U with the same marked length spectrum as g_0 is strongly isometric to g_0.*

One may also consider a similar question where we replace the function L_{g_0} by the *unmarked length spectrum*, that is, we just look at the *set* of lengths of closed geodesics on M, viewed as a subset of $(0, \infty)$ with no additional structure. However, this question has a negative answer, already for closed hyperbolic surfaces. The first examples of non-isometric hyperbolic surfaces with the same unmarked length spectrum are due to Vigneras VIGNÉRAS (1980).

The goal of this survey is to give a short historical account on partial results towards the marked length spectrum conjecture and to outline the main steps of the proof of Theorem 1.2, giving a more detailed explanation of its assumptions along the way.

2. Earlier results towards the marked length spectrum conjecture

Nonpositively curved Riemannian metrics on closed oriented surfaces of genus $h \geqslant 2$ have always been considered as a test case for the understanding of negatively curved metrics on manifolds of all dimensions, although the analogy is problematic due to the fact that by uniformization, any smooth metric g on such a surface S is *conformally equivalent* to a hyperbolic metric. That is, there is a smooth function ρ on S so that the metric $e^{\rho} g$ is of constant curvature -1. This gives strong additional constraints which do not exist in higher dimension.

In contrast to hyperbolic metrics on closed manifolds of dimension at least 3, a hyperbolic metric on a surface S of genus $h \geqslant 2$ is not unique up to isometry: There is an entire moduli space of isometry classes of hyperbolic metrics on S of dimension $6h - 6$. Such hyperbolic metrics can be constructed explicitly, and there is a collection of $6h - 5$ conjugacy classes of *simple* closed curves on S, that is, curves without self-intersection, whose lengths completely determine the hyperbolic metric SCHMUTZ (1993).

Understanding the marked length spectrum of a negatively curved metric g on S in a fixed conformal class is already interesting. The corresponding rigidity question was answered affirmatively by Katok KATOK (1988). His argument immediately extends to the following

Theorem 2.1 (Katok). *Let g, g_0 be two smooth conformally equivalent Riemannian metrics of negative curvature on a closed manifold M of dimension $n \geqslant 2$. If g, g_0 have the same marked length spectrum then they are isometric.*

The proof of this result is quite short. We present a sketch as it rests on two basic principles which are important cornerstones for later progress. For this and for later use, define the *geodesic flow* Φ^t on the unit tangent bundle $P : T^1M \to M$ of a Riemannian manifold (M, g) by $\Phi^t v = \gamma_v'(t)$ where γ_v is the geodesic with initial velocity v. The flow Φ^t preserves the *Lebesgue Liouville measure* μ, which is locally defined by a smooth volume form on T^1M whose integration over the fibers of the bundle T^1M equals the volume element of the metric g on M. Periodic orbits of Φ^t are precisely the unit tangent lines of closed geodesics.

If M is closed and the metric on M is negatively curved, then the flow Φ^t is an *Anosov flow*: Let X be its generator. There exists a $d\Phi^t$-invariant decomposition

$$TT^1M = E^+ \oplus E^- \oplus \mathbb{R}X, \tag{1}$$

and there exists a number $\alpha > 0$ with

$$\|d\Phi^{\mp t} w\| \leqslant e^{-\alpha t}\|w\|$$

for every $w \in E^{\pm}$, with a suitable choice $\| \; \|$ of a norm on TT^1M defined by some Riemannian metric. The decomposition (1) is called the *Anosov splitting*. It is known to be Hölder continuous, but in general, it is not smooth.

The Anosov property for Φ^t has the following two consequences. First, the normalized Lebesgue Liouville measure $\hat{\mu} = \mu/\mu(T^1M)$ is *ergodic* for Φ^t. This means that whenever $A \subset T^1M$ is a Φ^t-invariant Borel set, then either $\hat{\mu}(A) = 0$ or $\hat{\mu}(T^1M - A) = 0$. In particular, by the *Birkhoff ergodic theorem*, for any L^2-integrable function f on T^1M and for $\hat{\mu}$-almost every $v \in T^1M$, we have

$$\int f d\hat{\mu} = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\Phi^s v) ds.$$

Here the existence of the limit on the right hand side of this equation is part of the statement of the theorem.

Furthermore, the following *Anosov closing lemma* holds true. Let d be any distance function on T^1M defined by a Riemannian metric. Then for any $\delta > 0$, there are numbers $\varepsilon = \varepsilon(\delta) > 0$, and $T_0 = T_0(\delta) > 0$ with the following property. If for some $v \in T^1M$ and some $T > T_0$, we have $d(v, \Phi^T v) < \varepsilon$, then there exists a periodic orbit η for Φ^t, of period $L(\eta) \in [T - \delta, T + \delta]$, such that $d(\Phi^t v, \eta(t)) < \delta$ for all $t \in [0, T]$.

Since continuous functions on compact spaces are uniformly continuous, one obtains as a consequence of the Birkhoff ergodic theorem and the Anosov closing lemma the following.

Corollary 2.2. *Let $f : T^1M \to \mathbb{R}$ be a continuous function. Then for every $\varepsilon > 0$ and $T_0 > 0$, there exists a periodic point v for Φ^t of period $T > T_0$ such that*

$$\left|\frac{1}{T}\int_0^T f(\Phi^t v) dt - \int f d\hat{\mu}\right| < \varepsilon.$$

Sketch of a proof of Theorem 2.1. Let g, g_0 be negatively curved metrics on the same closed manifold M such that $g = \rho g_0$ for a smooth function ρ on M. Assume that g, g_0 have the same marked length spectrum. By perhaps exchanging g and g_0 we may assume that $\mathrm{vol}(M, g) \leqslant \mathrm{vol}(M, g_0)$ (here vol denotes the volume).

Denote by $P : T^1M_0 \to M$ the unit tangent bundle of M for the metric g_0, equipped with the Lebesgue Liouville measure μ, and let ω be the volume element of g_0 on M. Then $\rho^{n/2}\omega$ is the volume element for the metric g on M and hence naturality under pull-back shows that

$$\int_{T^1M_0} (P^*\rho)^{n/2} d\mu = \mathrm{vol}(S^{n-1})\mathrm{vol}(M, g) = \mathrm{vol}(T^1M, g) \leqslant \mu(T^1M_0). \qquad (2)$$

On the other hand, the integral of the function $\rho^{1/2}$ over each closed geodesic γ for the metric g_0, parameterized by arc length, is the g-length of γ. As the marked length spectra of g and g_0 coincide, this length is not smaller than the g_0-length $L_{g_0}(\gamma)$ of γ. Thus if we denote by Φ^t the geodesic flow on T^1M_0, then for every periodic orbit η of Φ^t, we have

$$\int_\eta (P^*\rho)^{1/2} dt \geqslant L(\eta) \qquad (3)$$

where $L(\eta)$ is the period of the orbit (which is just the length of the corresponding closed geodesic for g_0).

Write $\hat{\mu} = \mu/\mu(T^1M_0)$. By Corollary 2.2, since the function $(P^*\rho)^{1/2}$ on T^1M_0 is continuous and fulfills the inequality (3) for all periodic orbits η for Φ^t, we have

$$\int_{T^1M_0} (P^*\rho)^{1/2} d\hat{\mu} \geqslant 1. \qquad (4)$$

Together with inequality (2), this shows that $\int_{T^1M_0} (P^*\rho)^{1/2} d\hat{\mu} \geqslant 1 \geqslant \int_{T^1M_0} (P^*\rho)^{n/2} d\hat{\mu}$. It now follows from the Hölder inequality that this is possible only if the function ρ is constant and hence if g, g_0 are isometric. □

The proof of Theorem 2.1 motivates the following extension of Conjecture 1.1.

Conjecture 2.3. *Let g, g_0 be two negatively curved metrics on a closed manifold M. If $L_g(c) \geqslant L_{g_0}(c)$ for each conjugacy class $c \in \mathscr{C}$, then* $\mathrm{vol}(M, g) \geqslant \mathrm{vol}(M, g_0)$, *with equality only if g, g_0 are strongly isometric.*

Shortly after the appearance of the article of Katok, Conjecture 1.1 for surfaces was settled by Otal Otal (1990) and, independently, Croke Croke (1990). They showed

Theorem 2.4 (Croke, Otal). *Let g, g_0 be two smooth nonpositively curved metrics on a closed surface of genus $h \geqslant 2$. If g, g_0 have the same marked length spectrum, then they are isometric.*

The approach of both authors is similar and rests on the following two facts. The first fact is valid in all dimensions.

Fact 1: If two metrics g, g_0 on a closed manifold M of dimension $n \geqslant 2$ are non-positively curved, have the same marked length spectrum and Anosov geodesic flows Φ^t on their unit tangent bundles T^1M, T^1M_0, then these geodesic flows are *time preserving conjugate*: There exists a Hölder continuous map $F : T^1M \to T^1M_0$ such that $\Phi^t \circ F = F \circ \Phi^t$. The map F gives information on the coupling of lengths of periodic orbits which are in a suitable sense close to each other.

Fact 2: For a closed surface S with non-positively curved metric g, one can reconstruct the Lebesgue Liouville measure on the unit tangent bundle T^1S of S from the marked length spectrum using the fact that for surfaces, geodesics which intersect transversely a given open geodesic segment in the universal covering $\tilde{S}$ of S form an open subset of the space of all geodesics on $\tilde{S}$ whose measure (for the projection to the space of geodesics on $\tilde{S}$ of the Liouville measure on the unit tangent bundle of $\tilde{S}$) equals π times the length of the segment. As a consequence, if two such metrics g, g_0 have the same marked length spectrum, then the time preserving conjugacy between their geodesic flows conjugates the Liouville measures for g, g_0, and the volumes of S with respect to g, g_0 coincide.

Embarking from these two facts, the proof of the marked length spectrum rigidity theorem for surfaces uses an ingenious and fairly elementary but purely 2-dimensional construction.

The only global result which is known in all dimensions is the following special case of Conjecture 1.1 Hamenstädt (1999).

Theorem 2.5 (Hamenstädt). *Let (M, g_0) be a closed rank 1 locally symmetric manifold. If g is another negatively curved metric on M with the same marked length spectrum as g_0, then the metrics g, g_0 are strongly isometric.*

The proof of this result consists of two independent steps. The first step resembles the approach for surfaces. Namely, it is shown that whenever g, g_0 are metrics on M with Anosov geodesic flow and such that the Anosov splitting for g_0 is of class C^1, and if the metrics have the same marked length spectrum, then the volumes of g, g_0 coincide. In fact, a time preserving conjugacy between the geodesic flows for g, g_0 maps the Lebesgue Liouville measure for g to the Lebesgue Liouville measure for g_0. Examples of metrics with C^1-Anosov splitting are locally symmetric metrics or metrics whose sectional curvature is strictly 1/4-pinched.

With this information, the rigidity statement follows from the following deep theorem of Besson, Courtois and Gallot Besson, Courtois, and Gallot (1995). For its formulation, define the *volume entropy* of a negatively curved metric on a closed manifold M to be the quantity

$$h_{\mathrm{vol}} = \lim_{R \to \infty} \frac{1}{R} \log \mathrm{vol}(B(x, R))$$

where $B(x, R)$ is the ball of radius R about x in the universal covering $\tilde{M}$ of M. The limit is known to exist and to be independent of the basepoint x.