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RECENT PROGRESSES ON THE SUBCONVEXITY PROBLEM

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The subconvexity problem aims at providing non-trivial (i.e. subconvex) bounds for central values of automorphic L-functions; the main conjecture in this area is the Generalized Lindelöf Hypothesis which itself is a consequence of the Generalized Riemann Hypothesis. This lecture will survey several advances that have been made on this question during the past ten years: these include, the delta-symbol approach of R. Munshi, the Weyl type bounds of I. Petrow and M. Young (both use the Dirichlet L-series representation of the central values) and the works of P. Nelson and A. Venkatesh (which use the automorphic period representations for the central value)

1. Introduction

The Riemann zeta function is initially defined as the converging series

$$\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s} = \prod_p \big(1 - \frac{1}{p^s}\big)^{-1}$$

for $\mathfrak{Re}\, s > 1$. As is well known it has an analytic continuation to $\mathbf{C}$ (with a simple pole at $s = 1$) and satisfies a functional equation relating its values at s and at $1 - s$. In particular the most mysterious region (from the analytic viewpoint at least) to study $\zeta(s)$ is the *critical strip* $0 \leqslant \mathfrak{Re}\, s \leqslant 1$.

One hundred years ago, Weyl (1921) introduced an important technique (now called the *Weyl differencing method*) to investigate the growth of the Riemann zeta function along the edge of the critical strip, that is $\zeta(1 + it)$ for $t \to \infty$.

During the same year, Hardy and Littlewood realized the potential of Weyl's method and announced strong *upper bounds* for $\zeta(s)$ for s *inside* the critical strip and in particular along its center, the *critical line* $\mathfrak{Re}\, s = 1/2$: using Weyl's method, they obtained the upper bound

$$\zeta(1/2 + it) = O(1 + |t|^{1/6}). \tag{1.1}$$

This bound improved significantly on Lindelöf's 1908 bound

$$\zeta(1/2+it) = O(1+|t|^{1/4}) \tag{1.2}$$

which was a consequence of the *Phragmen–Lindelöf convexity principle* (itself, a consequence of the maximum principle). Hardy and Littlewood did not publish their proof in details, but it should have been as follows: by their *approximate functional equation* formula for $\zeta(s)$ (published in 1927), one has for $|t| \geqslant 1$,

$$\zeta(1/2+it) = \sum_{n\leqslant(|t|/2\pi)^{1/2}} \frac{1}{n^{1/2+it}} + \pi^{1/2-s}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{n\leqslant(|t|/2\pi)^{1/2}} \frac{1}{n^{1/2-it}} + O(|t|^{-1/4}). \tag{1.3}$$

In particular, bounding all the terms in this sum trivially, one recover Lindelöf's bound (1.2) and going beyond amounts to detect further cancellations coming from the oscillations of the argument of $n^{-1/2\pm it}$, $n \leqslant (|t|/2\pi)^{1/2}$ when t is large. This is precisely what Weyl's method was able to capture and this eventually led to (1.1).

This so-called *Weyl bound* was the first example of a *subconvex bound* (because it improve a bound derived from a convexity principle) for the very first L-function.

The *Subconvexity Problem* is the general problem of obtaining *subconvex* bounds for the values of general L-functions along the critical line.

2. L-functions and the convexity bound

We will describe shortly the class of L-functions we will be considering but for the moment we will isolate the most basic properties they satisfy (or sometimes are expected to satisfy). In any case an L-function will be a non-zero Dirichlet series

$$L(\pi,s) = \sum_{n\geqslant 1} \frac{\lambda_\pi(n)}{n^s}$$

associated to an arithmetic function $\lambda_\pi\colon \mathbf{N}_{>0} \to \mathbf{C}$, absolutely converging for $\Re\mathfrak{e}\, s > 1$, coming with some additional data and enjoying (amongst others) the following analytic properties (see Iwaniec and Kowalski, 2004, §5.1)

1. *Euler product.* For $\Re\mathfrak{e}\, s > 1$, the serie $L(\pi,s)$ factors into an Euler product of *local L-factors* of degree $\leqslant d$: for $\Re\mathfrak{e}\, s > 1$,

   $$L(\pi,s) = \prod_p L_p(\pi,s),\ L_p(\pi,s) := \prod_{i=1}^{d}\Big(1-\frac{\alpha_{\pi,i}(p)}{p^s}\Big)^{-1},$$

   for p ranging over the set of prime number; the $\alpha_{\pi,i}(p)$, $i = 1,\cdots,d$ are complex numbers satisfying $|\alpha_{\pi,i}(p)| < p$. In particular the arithmetic function $n \mapsto \lambda_\pi(n)$ is multiplicative: $\lambda_\pi(1) = 1$ and $\lambda_\pi(mn) = \lambda_\pi(m)\lambda_\pi(n)$ if $(m,n) = 1$.

2. *Non-archimedean local parameters.* The multiset $\{\alpha_{\pi,i}(p),\ i=1,\cdots,d\}$ is called the set of *local parameters* of $L(\pi,s)$ at p and $L_p(\pi,s)$ is called the local factor at p. Moreover, there exists an integer $q(\pi)\geqslant 1$ (the *arithmetic conductor* of the L-function) such that if p does not divide $q(\pi)$
$$\left|\prod_{i=1}^{d}\alpha_{\pi,i}(p)\right|=1,$$
so that the local factor has degree d exactly. The primes p not dividing $q(\pi)$ are then called *unramified*.

3. *Archimedean local parameters.* This collection of non-archimedean local parameters is completed by a multiset of complex numbers, $\{\mu_{\pi,i},\ i=1,\cdots,d\}$ satisfying $\mathfrak{Re}\,\mu_{\pi,i}<1$ and called *the local parameters at* ∞; associated to it is a corresponding archimedean local factor which this time, is a product of Gamma functions
$$L_\infty(\pi,s)=\prod_{i=1}^{d}\Gamma_{\mathbf{R}}(s-\mu_{\pi,i}).$$

4. *Analytic continuation and functional equation*: so far $L(\pi,s)$ was essentially specified by a collection of local factors $L_p(\pi,s)$ which could be largely random. What qualifies it as an L-function is the following properties of global nature: the function $s\mapsto L(\pi,s)$ admits meromorphic continuation to $\mathbf{C}$ with at most finitely many poles. Moreover $L(\pi,s)$ satisfies a functional equation of the shape
$$\Lambda(\pi,s)=\varepsilon(\pi)\overline{\Lambda(\pi,1-\overline{s})}$$
where $\varepsilon(\pi)$ (the root number) is a complex number of modulus 1, and $\Lambda(\pi,s)$ (the *completed L-function*) is given by
$$\Lambda(\pi,s):=q(\pi)^{s/2}L_\infty(\pi,s).L(\pi,s)$$
for $q(\pi)\geqslant 1$ the arithmetic conductor mentioned above. The pole of the completed L-function are located on the vertical lines $\mathfrak{Re}\,s=0,1$ and the sum of their orders is bounded by $\leqslant 2d$ and outside of these poles, $\Lambda(\pi,s)$ has rapid decay in any bounded vertical strip $\{s,\ A\leqslant\mathfrak{Re}\,s\leqslant B\}$.

Remark 1. In particular the *dual* Dirichlet series given by
$$L(\pi^\vee,s):=\overline{L(\pi,\overline{s})}=\sum_{n\geqslant 1}\frac{\overline{\lambda_\pi(n)}}{n^s},\ \mathfrak{Re}\,s>1$$
qualify as an L-function with $q(\pi^\vee)=q(\pi)$.

2.1. The Convexity Bound

Given $L(\pi, s)$ an L-function as above; we would like to evaluate the growth of $L(\pi, 1/2 + it)$ as $t \to \infty$. Since for $\mathfrak{Re}\, s > 1$, $L(\pi, s)$ is given by a converging Euler product, we expect and often understand "well" the analytic behaviour of $L(\pi, s)$ in this region (for instance $L(\pi, s)$ has no zeros there); in particular for any $\varepsilon > 0$, we have

$$L(\pi, 1 + \varepsilon + it) \ll_{d,\varepsilon} 1.$$

By the functional equation (and the known properties of the Gamma function) we then expect and often understand "well" the behaviour of $L(\pi, s)$ when $\mathfrak{Re}\, s < 0$; by Stirling's formula, the previous bound implies that for t large enough

$$L(\pi, -\varepsilon + it) \ll_{\varepsilon} |t|^{(1+\varepsilon)\frac{d}{2}}.$$

For σ in the interval $[-\varepsilon, 1 + \varepsilon]$, the convexity principle (see Iwaniec and Kowalski, 2004, Chap. 5, A.2) then implies that $L(\pi, s)$ is bounded by the convex multiplicative combination of the bounds at the extremities:

$$L(\pi, \sigma + it) \ll |t|^{\frac{d}{2}(1-\sigma+O(\varepsilon))}$$

and for $\sigma = 1/2$ one obtains (in the s variable)

$$L(\pi, 1/2 + it) \ll_{\varepsilon} |t|^{\frac{d}{4}+\varepsilon}.$$

In this bound we have ignored the other quantities on which $L(\pi, s)$ might depend: the conductor and the spectral parameter. The above argument can be refined to take these into account by introducing the *analytic conductor* of $L(\pi, s)$: it is defined (in a ad-hoc way) for $s = 1/2 + it$ by

$$Q(\pi, s) = q(\pi) \prod_{i=1}^{d} (1 + |\mu_{\pi,i} - it|) = q(\pi) q_{\infty}(\pi, s);$$

also to simplify notations we will write

$$Q(\pi) = Q(\pi, 1/2),\ q_{\infty}(\pi) = q_{\infty}(\pi, 1/2) = \prod_{i=1}^{d} (1 + |\mu_{\pi,i}|).$$

With suitable additional assumption on $L(\pi, s)$, one can obtain the

Convexity Bound. *Let $L(\pi, s)$ be an L-function of degree $d \geqslant 1$, for any $\varepsilon > 0$ and $s = \frac{1}{2} + it$, $i \in \mathbf{R}$, one has*

$$L(\pi, s) \ll_{d,\varepsilon} Q(\pi, s)^{1/4+\varepsilon}. \tag{2.1}$$

We will give here an alternative proof similar to that given in the introduction: for this we need a modern form of the approximate functional equation (2.2). By an appropriate Mellin transformation, a contour shift and the functional equation, one can show that (Iwaniec and Kowalski, 2004, Thm 5.3 & Prop. 5.4):

Approximate Functional Equation. *Let $L(\pi, s)$ be an L-function satisfying the analytic properties above. There exist two smooth functions*

$$V_s, V_{1-s}\colon \mathbf{R}_{>0} \to \mathbf{C}$$

whose derivatives have rapid decay: for any $y > 0$, any integer $a \geqslant 0$ and any $A > 0$ one has

$$y^a V_\bullet^{(a)}(y) \ll_{d,A,a,\mathfrak{Re}\, s} (1+y)^{-A}$$

(*although these functions might depend on the archimedean parameters of π, the implicit constants depend only on d, A and a*) *such that*

$$L(\pi,s) = \sum_{n\geqslant 1} \frac{\lambda_\pi(n)}{n^s} V_s\Big(\frac{n}{Q(\pi,s)^{1/2}}\Big) + \varepsilon(\pi,s) \sum_{n\geqslant 1} \frac{\overline{\lambda_\pi}(n)}{n^{1-s}} V_{1-s}\Big(\frac{n}{Q(\pi,s)^{1/2}}\Big) + R(\pi,s) \tag{2.2}$$

where $\varepsilon(\pi,s)$ is a complex number of modulus 1 *and $R(\pi,s)$ is a contribution from the poles of $\Lambda(\pi,s)$ and is zero if $\Lambda(\pi,s)$ is entire.*

Proof of the convexity bound. We sketch the proof (in a slightly stronger form) assuming that $L(\pi,s)$ is entire and that its local parameters satisfy the following Ramanujan–Peterson type bound

$$\forall p,\ i = 1,\dots,d,\ |\alpha_{\pi,i}(p)| \leqslant 1.$$

In particular the coefficients $\lambda_\pi(n)$ are bounded by

$$|\lambda_\pi(n)| \leqslant \tau_d(n) = \sum_{n_1.\cdots.n_d=n} 1$$

the d-th order divisor function. By the approximate functional equation we have taking $A \geqslant 2$

$$L(\pi,s) \ll_{d,A} \sum_{n\geqslant 1} \frac{\tau_d(n)}{n^{1/2}} \Big(1+\frac{n}{Q(\pi,s)^{1/2}}\Big)^{-A} \ll_d Q(\pi,s)^{1/4} \log^{d-1}(Q(\pi,s)). \qquad \square$$

Remark 2. While the convexity bound is trivial to prove in favourable cases such that this one here, it is not obvious in general (see Molteni, 2002 and Brumley, 2004).