Séminaire BOURBAKI Janvier 2022
73e année, 2021–2022, no 1191, p. 403 à 432
doi: 10.24033/ast.1192

FINITE TIME BLOW UP FOR THE COMPRESSIBLE FLUIDS AND FOR THE ENERGY SUPERCRITICAL DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION
[*after* **Frank Merle, Pierre Raphaël, Igor Rodnianski and Jérémie Szeftel**]

by **Galina Perelman**

Introduction

The problem of finite time breakdown of solutions starting from smooth initial data is one of the central problems in the theory of nonlinear evolution PDEs. In this talk we will address this problem in the context of the following two models: the isentropic compressible Navier-Stokes equation and its inviscid Euler limit on the one hand and the defocusing nonlinear Schrödinger equation on the other hand. The aim of the talk is to report on breakthrough progress recently made in a series of works of F. Merle, P. Raphaël, I. Rodnianski and J. Szeftel who showed that both models in a suitable range of parameters, admit a finite time blow up regime governed by appropriate self-similar solutions of the underlying Euler equation. We start by briefly overviewing the history of the blow up problem for each of these models and explaining the connection between them.

The motion of isentropic compressible viscous fluids in $\mathbb{R}^d$ is governed by the compressible Navier-Stokes equations:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \rho\partial_t v + \rho v\cdot\nabla v + \nabla P(\rho) = \mu\Delta v + \mu'\nabla \operatorname{div} v \\ (\rho, v)|_{t=0} = (\rho_0, v_0), \end{cases} \tag{1}$$

where $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field, $\rho : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is the density of the fluid, μ, μ' are viscosity coefficients satisfying $\mu \geqslant 0$, $\mu' \geqslant \left(1-\frac{2}{d}\right)\mu$ and $P = P(\rho)$ is the pressure that we will assume to be given by:

$$P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma, \quad \gamma > 1. \tag{2}$$

In the inviscid limit $\mu = \mu' = 0$ one obtains the compressible Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho \partial_t v + \rho v \cdot \nabla v + \nabla P(\rho) = 0 \\ (\rho, v)|_{t=0} = (\rho_0, v_0). \end{cases} \tag{3}$$

We will be interested in solutions (ρ, v) that decay to zero at spatial infinity[(1)] keeping the density strictly positive:

$$\lim_{|x|\to\infty} (\rho(t,x), v(t,x)) = 0, \quad \rho(t,x) > 0, \tag{4}$$

and will focus mainly on the 3d case.

Solutions to (1) satisfy formally the mass and momentum conservation law

$$\int_{\mathbb{R}^d} \rho(t) dx = \int_{\mathbb{R}^d} \rho_0 dx, \quad \int_{\mathbb{R}^d} \rho(t) v(t) dx = \int_{\mathbb{R}^d} \rho_0 v_0 dx,$$

and the energy identity

$$\int_{\mathbb{R}^d} \left(\frac{1}{2} \rho(t) |v(t)|^2 + \frac{1}{\gamma} \rho^\gamma(t) \right) dx + \int_0^t (\mu \|\nabla v(s)\|^2_{L^2(\mathbb{R}^d)} + \mu' \|\operatorname{div} v(s)\|^2_{L^2(\mathbb{R}^d)}) ds$$
$$= \int_{\mathbb{R}^d} \left(\frac{1}{2} \rho_0 |v_0|^2 + \frac{1}{\gamma} \rho_0^\gamma \right) dx.$$

Note also that the Navier-Stokes equation (1) is preserved by the scaling

$$(\rho(t,x), v(t,x)) \mapsto (\lambda^{\frac{2(\mathrm{r}-1)}{\gamma-1}} \rho(\lambda^{\mathrm{r}} t, \lambda x), \lambda^{\mathrm{r}-1} v(\lambda^{\mathrm{r}} t, \lambda x)), \ \lambda > 0, \tag{5}$$

with $\mathrm{r} = \frac{2\gamma}{\gamma+1}$. The Euler equation (3) is invariant with respect to transformations (5) for any r.

Smooth, suitably decaying initial data (ρ_0, u_0) with strictly positive density are known to give rise to unique local in time strong solutions to (1), (3) (see Section 1 for the precise statements and references), that however do not always exist for all times, the conservation laws being far too weak to prevent the formation of singularities. Finite time breakdown of strong solutions to (1) starting from initial data with non-vanishing density, non-vanishing momentum and with suitable decay at infinity was shown by Rozanova (2008) in the case of $d \geqslant 3$, $\gamma \geqslant \frac{2d}{d+2}$, see also Xin (1998) where the case of non-barotropic compressible Navier-Stokes equations with compactly supported initial data was considered. For the 3d Euler equation (3)

(1) For the Euler equation the behavior at infinity is less important because of the domain of dependence principle.

the corresponding results go back to the work of Sideris (1985) who exhibited an open set of smooth initial data corresponding to compactly supported perturbations of constant states, including arbitrary small disturbances, that lead to classical solutions with a finite lifespan. However the proofs of Rozanova (2008), Sideris (1985), and Xin (1998), being based on convexity type arguments give no information on the nature of the singularity that develops.

For the compressible Euler equations, the typical singularity (at least for "small" initial data) is a shock[(2)]. In dimension one, the fact that initially smooth solutions can form shock singularities even when the initial data are small and compactly supported perturbations of a constant state is known since the works of Riemann. We refer to the monographs Dafermos (2010) and Majda (1984) for the details and references of the 1d theory which by now is quite complete. An important advance in understanding of multidimensional shock formation was achieved by Alinhac (1999, 2001), who considered a general class of quasilinear wave equations in dimensions two and three, including the irrotational compressible Euler equations, and showed that the failure of the Klainerman null condition in the equation leads for non-degenerate small compactly supported initial data to finite time shock formation caused by the crossing of characteristics (see also the precursor work of John, 1985). While giving a detailed description of the solutions up to the first singular time, the results of Alinhac (1999, 2001) leave open a more general question of the maximal smooth development of the initial data. For the 3d relativistic Euler equations, the latter was studied in the seminal work of Christodoulou (2007), see also Christodoulou and Miao (2014) for the non-relativistic case. The results of Christodoulou (2007) and Christodoulou and Miao (2014) cover the case of small compactly supported initial perturbations of constant state solutions, showing shock formation in irrotational space-time regions and giving a precise description of the corresponding portion of the boundary of the maximal classical development of the data. We also refer to the works of Buckmaster, Drivas, et al. (2021), Buckmaster, Shkoller, and Vicol (2019a,b, 2020), Christodoulou (2019), and Luk and Speck (2018, 2021) for further developments in the study of shock formation for the compressible Euler equations, including the results going beyond the irrotational and isentropic regimes.

[(2)] Shock singularity means that the velocity and density remain bounded while some of their first order derivatives blow up.

Shocks are not the only possible singularities for (3). Stronger singularities with both the density and the velocity blowing up, may occur as well. It has been known since the works of GUDERLEY (1942) and SEDOV (1959) that (3) has a family of spherically symmetric self-similar solutions

$$\rho(t,x)=\frac{1}{(T-t)^{\frac{2(\mathrm{r}-1)}{\mathrm{r}(\gamma-1)}}}\mathscr{R}\left(\frac{x}{(T-t)^{\frac{1}{\mathrm{r}}}}\right),\ v(t,x)=\frac{1}{(T-t)^{1-\frac{1}{\mathrm{r}}}}\mathscr{V}\left(\frac{x}{(T-t)^{\frac{1}{\mathrm{r}}}}\right). \quad (6)$$

Although typically these solutions are either non global or non-smooth (that is the profiles $\mathscr{R}$ and $\mathscr{V}$ are non-smooth), MERLE, RAPHAËL, RODNIANSKI, and SZEFTEL (2022b) proved that in a suitable range of parameter γ, and for a suitable sequence of blow up rates r, (3) admits global, decaying at infinity, C^∞ self-similar solutions. Furthermore, MERLE, RAPHAËL, RODNIANSKI, and SZEFTEL (2022c) showed that these C^∞ self-similar solutions can be used as a leading order approximation to generate finite energy(3) blow up solutions for both the Euler equation (3) and the Navier-Stokes equation (1). For the Navier-Stokes equation this gives the first result with a complete description of singularity formation. The C^∞ smoothness of the self-similar profiles plays a crucial role in the analysis of MERLE, RAPHAËL, RODNIANSKI, and SZEFTEL (2022c).

What is even more remarkable is that the above self-similar eulerian solutions can be also used to produce finite time blow up solutions for the defocusing nonlinear Schrödinger equation (NLS):

$$\begin{cases} iu_t=-\Delta u+|u|^{2p}u, \quad x\in\mathbb{R}^d, \quad p>0\\ u|_{t=0}=u_0\in H^s(\mathbb{R}^d). \end{cases} \quad (7)$$

The term "defocusing" refers to the sign "+" before the nonlinearity.

The NLS equation (7) is invariant with respect to the scaling:

$$u(t,x)\mapsto \lambda^{\frac{1}{p}}u(\lambda^2 t,\lambda x), \quad \lambda>0, \quad (8)$$

which preserves the homogeneous Sobolev norm $\|u_0\|_{\dot H^{s_c}(\mathbb{R}^d)}$ with $s_c=\frac{d}{2}-\frac{1}{p}$.

Local well-posedness of (7) is classical and goes back to the works of GINIBRE and VELO (1979). The Cauchy problem (7) is known to be locally well-posed in H^s for(4) $s\geqslant \max\{0,s_c\}$ (see e.g. CAZENAVE (2003) and CAZENAVE and WEISSLER (1990) and

(3) Although decaying at infinity, these self-similar solutions have infinite energy, see Section 3.
(4) In the case when p is not an integer one has also to assume that s is compatible with the smoothness of the nonlinear term.

references therein). For $s \geqslant \max\{1, s_c\}$, the solutions satisfy on their lifespan the mass and energy conservation laws:

$$M(u(t)) \equiv \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0),$$

$$E(u(t)) \equiv \int_{\mathbb{R}^d} \left(|\nabla u(t,x)|^2 + \frac{1}{p+1} |u(t,x)|^{2p+2} \right) dx = E(u_0).$$

In the case $s > s_c$ the lifespan of the solutions admits a lower bound depending only on the H^s norm of initial data[(5)], which in a standard way, implies that the solution of (7) is either global or its H^s norm becomes unbounded in finite time. By the mass and energy conservation, this ensures global well-posedness in H^s, $s \geqslant 1$, in the energy subcritical case $s_c < 1$. Global well-posedness is known to persist in the energy critical case $s_c = 1$ ($p = \frac{2}{d-2}, d \geqslant 3$). This was proved (after considerable efforts) by Bourgain (1999), Grillakis (2000), Tao (2005) for spherically symmetric initial data, and by Colliander et al. (2008), Ryckman and Visan (2007), and Visan (2007) for general data. We also refer to the seminal paper of Kenig and Merle (2006) where the powerful technology of concentration compactness/rigidity method was introduced.

The question whether finite time blow up occurs in the energy supercritical case $s_c > 1$ ($p > \frac{2}{d-2}, d \geqslant 3$) remained completely open for long time. On the one hand, numerical simulations as well as the global well-posedness results for the log-supercritical equations (see e.g. Tao, 2007), the nonexistence of soliton like solutions and the expected nonexistence of the self-similar blow up supported the hypothesis of global well-posedness. On the other hand, Tao (2018) exhibited examples of energy supercritical defocusing NLS systems for which finite time blow up does happen.

A decisive breakthrough has been achieved by Merle, Raphaël, Rodnianski, and Szeftel (2022a) who considered the energy supercritical NLS

$$iu_t = -\Delta u + |u|^{2p} u, \quad x \in \mathbb{R}^d, \quad p > \frac{2}{d-2} \tag{9}$$

in dimensions $5 \leqslant d \leqslant 9$ and showed that there exist, for certain choices of p, C^∞ well localized initial data leading to solutions blowing up in finite type. The construction of Merle, Raphaël, Rodnianski, and Szeftel (2022a) relies on the hydrodynamic formulation of the NLS equation (9) arising via the Madelung transform $u = \rho e^{i\varphi}$ that allows to view (9), at least in some regimes, as a perturbation of the compressible Euler equation (3) and to use the C^∞ self-similar solutions of the latter to produce finite time blow up solutions to (9).

[(5)] In fact, one has a slightly stronger result including the persistence of regularity: if $u_0 \in H^{s'}$ with $s' > s$, then the solution stays in $H^{s'}$ as long as it exists in H^s.