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THE GEOMETRIC DYNAMICAL NORTHCOTT PROPERTY FOR REGULAR POLYNOMIAL AUTOMORPHISMS OF THE AFFINE PLANE

by Thomas Gauthier & Gabriel Vigny

Abstract. — We establish the finiteness of periodic points, which we call the *geometric dynamical Northcott property*, for regular polynomials' automorphisms of the affine plane over a function field $\mathbf{K}$ of characteristic zero, improving results of Ingram.

To this end, we show that when $\mathbf{K}$ is the field of rational functions of a smooth complex projective curve, the canonical height of a subvariety is the mass of an appropriate bifurcation current and that a marked point is stable if and only if its canonical height is zero. We then establish the geometric dynamical Northcott property using a similarity argument.

Résumé (*La propriété dynamique géométrique de Northcott pour les automorphismes polynomiaux réguliers du plan affine*). — Nous établissons la finitude des points périodiques, ce que nous appelons la *propriété dynamique géométrique de Northcott* pour les automorphismes polynomiaux réguliers du plan affine sur un corps de fonctions $\mathbf{K}$ de caractéristique zéro, améliorant des résultats d'Ingram.

Pour cela, nous montrons que quand $\mathbf{K}$ est le corps des fonctions rationnelles sur une courbe projective lisse, la hauteur canonique d'une sous-variété est la masse du courant de bifurcation approprié et qu'un point marqué est stable si et seulement si sa hauteur canonique est zéro. Nous établissons alors la propriété dynamique géométrique de Northcott en utilisant un argument de similarité.

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1. Introduction

Let k be an algebraically closed field of characteristic zero, $\mathscr{B}$ a normal projective k-variety, and $\mathbf{K} := k(\mathscr{B})$ its field of rational functions. A regular plane automorphism over the function field $\mathbf{K}$ is a polynomial automorphism of the affine plane $\mathbb{A}^2_{\mathbf{K}}$, such that the unique indeterminacy point I^+ of its extension to $\mathbb{P}^2_{\mathbf{K}}$ is distinct to the unique indeterminacy point I^- of the extension of f^{-1} to $\mathbb{P}^2_{\mathbf{K}}$.

Let $h : \mathbb{P}^2(\bar{\mathbf{K}}) \to \mathbb{R}_+$ be the standard height function on $\mathbb{P}^2(\bar{\mathbf{K}})$, i.e. the height function $h = h_{\mathbb{P}^2,L}$ associated with the ample line bundle $L := \mathscr{O}_{\mathbb{P}^2}(1)$. Following Kawaguchi [15] in the number field case, one can define three different canonical heights for f:

$$\widehat{h}_f^+ := \lim_{n\to+\infty} d^{-n} h\circ f^n, \quad \widehat{h}_f^- := \lim_{n\to+\infty} d^{-n} h\circ f^{-n} \quad \text{and} \quad \widehat{h}_f := \widehat{h}_f^+ + \widehat{h}_f^-,$$

where d is the common degree of f and f^{-1}. The height function $\widehat{h}_f^+$ (or $\widehat{h}_f^-$) detects the arithmetic complexity of the forward orbit (or of the backward orbit) of a point in $\mathbb{A}^2(\bar{\mathbf{K}})$.

A particularly interesting case of regular plane automorphisms is Hénon maps, i.e. maps of the form $f(x,y) = (ay, x + p(y))$ with $a \in \mathbf{K}^*$ and $p(x) \in \mathbf{K}[x]$. In that setting, Ingram proved the following ([14, Theorem 1.2])

Theorem 1.1 (Ingram). — *Let k be any field and let $\mathbf{K}$ be the field of rational functions of a smooth projective k-variety. Let $f(x,y) = (y, x + p(y))$ for $p(x) \in \mathbf{K}[x]$ of degree at least 2. Then either f is isotrivial, or else the set of elements $z \in \mathbb{A}^2(\mathbf{K})$ with $\widehat{h}_f(z) = 0$, is finite, bounded in size in terms of the number of places of bad reduction for f. In particular, if f is not isotrivial, then $\widehat{h}_f(z) = 0$ if and only if z is periodic for f.*

The map f is isotrivial if, after a suitable change of coordinates, the coefficients of f are constant, i.e. belong to k. Ingram asked whether one can prove a similar statement for $f(x,y) = (ay, x + p(y))$ with $a \in \mathbf{K}^*$. This is the purpose of this article in the case where k has characteristic zero (Ingram's result allows a positive characteristic). More precisely,

1. We generalize the above statement to any polynomial automorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ with dynamical degree $\lambda_f > 1$ defined over a function field of characteristic 0. More precisely, we establish the *geometric dynamical Northcott property*: if the map f is not isotrivial, then $\widehat{h}_f(z) = 0$ if and only if z is periodic for f, and there are only finitely many such points.
2. We replace the hypothesis that $\widehat{h}_f(P) = 0$ with the a priori weaker one $\widehat{h}_f^+(P) = 0$.
3. We express, in the case where $\mathbf{K}$ is the field of rational functions of a smooth complex projective curve, the canonical height of a subvariety as the mass of an appropriate bifurcation current. Then, we show that

a marked point is stable, in the sense of complex dynamics, if and only if its canonical height is zero.

Consider a plane automorphism f over the function field $\mathbf{K}$. Following the arguments of [6, section 3], it is enough to consider the case $k = \mathbb{C}$ and $\mathbf{K} = \mathbb{C}(\mathcal{B})$, which is what we do in the rest of this paper. This will enable the use of complex methods.

To a (regular) automorphism $f : \mathbb{A}^2_{\mathbf{K}} \to \mathbb{A}^2_{\mathbf{K}}$ we can associate a *model* over $\mathcal{B}$, where $\mathbf{K} = \mathbb{C}(\mathcal{B})$, i.e. a birational map $\mathfrak{f} : \mathbb{A}^2(\mathbb{C}) \times \mathcal{B} \dashrightarrow (f_\lambda(z), \lambda) \in \mathbb{A}^2(\mathbb{C}) \times \mathcal{B}$, such that $\pi \circ \mathfrak{f} = \pi$, where $\pi : \mathbb{A}^2(\mathbb{C}) \times \mathcal{B} \to \mathcal{B}$ is the canonical projection, and such that there exists a Zariski open subset $\Lambda \subset \mathcal{B}$, for which $\mathfrak{f}$ is restricted to $\mathbb{A}^2(\mathbb{C}) \times \Lambda$ as an automorphism, and such that $f_\lambda := \mathfrak{f}|_{\mathbb{A}^2(\mathbb{C}) \times \{\lambda\}}$ is a complex (regular) polynomial automorphism for any $\lambda \in \Lambda$. The map f can be identified with the restriction of $\mathfrak{f}$ to the generic fibre of π. The open set Λ is the *regular part* of the family $\mathfrak{f}$. To any point $z \in \mathbb{A}^2(\mathbf{K})$ one can also associate a rational map $\mathfrak{z} : \mathcal{B} \to \mathbb{C}$, such that $\mathfrak{z}$ is defined on Λ. Such a $\mathfrak{z}$ is called a *marked point.* We say that $\mathfrak{z}$ is *stable*, if the sequence of iterates $\lambda \mapsto f^n_\lambda(\mathfrak{z}(\lambda))$ is normal on compact subsets of Λ (see Remark 3.4 below).

Finally, we say that a polynomial automorphism $f : \mathbb{A}^2_{\mathbf{K}} \to \mathbb{A}^2_{\mathbf{K}}$ is *isotrivial*, if there exists an affine automorphism $\varphi : \mathbb{A}^2_{\overline{\mathbf{K}}} \to \mathbb{A}^2_{\overline{\mathbf{K}}}$, such that $\varphi^{-1} \circ f \circ \varphi$ is defined over $\mathbb{C}$, or equivalently if for any model $\mathfrak{f} : \mathbb{C}^2 \times \mathcal{B} \dashrightarrow \mathbb{C}^2 \times \mathcal{B}$ with regular part Λ and for any $\lambda, \lambda' \in \Lambda$, there is an affine automorphism $\varphi_{\lambda,\lambda'} : \mathbb{C}^2 \to \mathbb{C}^2$, such that $\varphi^{-1}_{\lambda,\lambda'} \circ f_\lambda \circ \varphi_{\lambda,\lambda'} = f_{\lambda'}$.

Our result can then be stated as

Main Theorem. — *Let $\mathfrak{f} : \mathbb{C}^2 \times \mathcal{B} \dashrightarrow \mathbb{C}^2 \times \mathcal{B}$ be a non-isotrivial algebraic family of regular polynomial automorphisms of degree $d \geq 2$ parametrized by a complex projective variety $\mathcal{B}$ with regular part Λ and let $f : \mathbb{A}^2_{\mathbf{K}} \to \mathbb{A}^2_{\mathbf{K}}$ be the induced regular automorphism over the field $\mathbf{K} = \mathbb{C}(\mathcal{B})$ of rational functions of $\mathcal{B}$. Then*

1. *for any point $z \in \mathbb{A}^2(\mathbf{K})$ with corresponding rational map $\mathfrak{z} : \mathcal{B} \dashrightarrow \mathbb{P}^2$,*

$$(\star) \qquad (\mathfrak{f}, \mathfrak{z}) \text{ is stable} \iff \widehat{h}_f(z) = 0 \iff \widehat{h}^+_f(z) = 0 \iff z \text{ is periodic.}$$

2. *The set of marked points $\{\mathfrak{z}$ such that $(\mathfrak{f}, \mathfrak{z})$ is stable$\}$ is a finite set. In particular, a stable marked point is stably periodic.*

We deduce the following corollary, which generalizes Ingram's theorem 1.1. As observed by the referee, it can be stated in the more general frame of automorphisms of $\mathbb{A}^2_{\mathbf{K}}$ of dynamical degree $\lambda_g > 1$ (see section 4.2 for the definitions of the dynamical degree and height in this setting).

Corollary. — *Let k be an algebraically closed field of characteristic zero and $\mathbf{K}$ be the field of rational functions of a projective k-variety. Let $g : \mathbb{A}^2_{\mathbf{K}} \to \mathbb{A}^2_{\mathbf{K}}$*

be a polynomial automorphism of dynamical degree $\lambda_g \geq 2$. Then either g is isotrivial or else the set of elements $P \in \mathbb{A}^2(\mathbf{K})$ with $\widehat{h}_g(P) = 0$ is finite.

In particular, if g is not isotrivial, then $\widehat{h}_g(P) = 0$ if and only if P is periodic for g.

For Hénon maps over number fields, the finiteness of periodic points is due to Silverman [21]. Constructing the canonical heights, Kawaguchi [15] proved this result over number fields for regular polynomial automorphisms using the classical Northcott property (see also Lee [19] for an alternate construction). Over a function field, such a result was established for polynomials of $\mathbb{A}^1$ by Benedetto [5] and rational maps of $\mathbb{P}^1$ by Baker [1] and DeMarco [9]. In higher dimension, Chatzidakis and Hrushovski gave a model-theoretic version of the statement for polarized endomorphisms in [7]. Finally, in [13], we extended the Northcott property to any polarized endomorphisms, giving a similar statement as that of the Main Theorem.

In order to prove the Main Theorem, we adapt the strategy of the proof of [13, Theorem A] to the case of regular polynomial automorphisms. New difficulties appear since we need to deal with indeterminacy points and saddle periodic points instead of repelling periodic points. Note that

- if f is non-isotrivial and $\lambda_0 \in \Lambda$ is fixed, then the set of parameters λ such that f_λ is conjugated to f_{λ_0} is a closed subvariety of Λ.
- The restriction of a stable marked point to a subvariety $\mathcal{B}'$ of $\mathcal{B}$ is still stable.
- If $\mathcal{B}'$ is a subvariety of $\mathcal{B}$ and $z \in \mathbb{C}(\mathcal{B})$ has height zero, then it defines a point in $\mathbb{C}(\mathcal{B}')$ whose height is again 0 by Bézout. Similarly, if for all $\mathcal{B}'$, the corresponding point in $\mathbb{C}(\mathcal{B}')$ has height 0, then so does z.

In particular, we can reduce this to the case where $\dim \mathcal{B} = 1$ by a slicing argument. We thus restrict ourselves to the case where $\mathbf{K}$ is the field of rational functions of a smooth complex projective curve $\mathcal{B}$. Finally, up to taking a branched cover of $\mathcal{B}$, conjugating by a suitable affine automorphism $\Phi \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{K}})$ and reducing Λ, we can assume that the indeterminacy point $I(f_\lambda)$ of f_λ is $[1:0:0]$, and the indeterminacy point $I(f_\lambda^{-1})$ of f_λ^{-1} is $[0:1:0]$ for every $\lambda \in \Lambda$.

Pick a point $z \in \mathbb{A}^2(\mathbf{K})$ and let $\mathcal{Z}_n$ be the irreducible subvariety of $\mathbb{P}^2(\mathbb{C}) \times \mathcal{B}$ induced by $f^n(z)$. Using Kawaguchi's comparison result on heights [15], we show that there exists some $B_0 > 0$, independent of z, such that $\widehat{h}_f(z) = 0$ implies $\deg_{\mathcal{M}}(\mathcal{Z}_n) \leq B_0$ for all $n \in \mathbb{Z}$, so there are only finitely many degrees to consider.

Let $\widehat{\omega} := \pi_{\mathbb{P}^2}^{-1}(\omega_{FS})$, where ω_{FS} is the Fubini–Study form on $\mathbb{P}^2(\mathbb{C})$ and $\pi_{\mathbb{P}^2} : \mathbb{P}^2 \times \mathcal{B} \to \mathbb{P}^2$ is the canonical projection. The *fibred Green current* of f is the positive closed $(1,1)$-current $\widehat{T}_{f^\pm}$ on $\mathbb{P}^2(\mathbb{C}) \times \Lambda$ defined by $\widehat{T}_{f^\pm} := \lim_{n\to+\infty} d^{-n}(f^{\pm n})^*(\widehat{\omega})$. As $\widehat{T}_{f^\pm}$ has continuous potential on $\mathbb{A}^2(\mathbb{C}) \times \Lambda$ and $\mathcal{Z} \cap \left(\mathbb{P}^2(\mathbb{C}) \times \Lambda\right) \subset \mathbb{A}^2(\mathbb{C}) \times \lambda$, the wedge product $[\mathcal{Z}] \wedge \widehat{T}_{f^\pm}$ is a well-defined measure ($[\mathscr{Z}]$ is the current of integration on $\mathscr{Z}$), which vanishes if and only if the marked point corresponding to z is stable (see remark 3.4). We show in Propositions 3.7 and 3.8 that

$$\widehat{h}_f^+(z) = \int_{\mathbb{C}^2\times\Lambda} [\mathscr{Z}] \wedge \widehat{T}_f \quad \text{and} \quad \widehat{h}_f^-(z) = \int_{\mathbb{C}^2\times\Lambda} [\mathscr{Z}] \wedge \widehat{T}_{f^{-1}}.$$

In particular, if $\mathcal{M}$ is an ample line bundle on $\mathbb{P}^2 \times \mathscr{B}$, $\widehat{h}_f^+(Z) = 0$ if and only if $(\deg_{\mathcal{M}}(\mathscr{Z}_n))_{n\geq 0}$ is bounded by some constant $D > 0$ independent of n if and only if $(\Lambda, f, \mathscr{Z})$ is (forward) stable. Hence, forward stability is, in fact, a global notion. To establish the above formulae, we prove a delicate degeneracy estimate of the Green function to deal with the indeterminacy point, which allows us to construct a DSH cut-off function (DSH functions, introduced by Dinh-Sibony [10], take into account the complex structure, whereas C^2 functions do not). Then we show that if $\mathscr{Z}$ is forward stable, then it is periodic (note that in [14], they do not relate zero height with stability). We use for that the Zariski density of saddle periodic points due to Bedford, Lyubich and Smillie [2].

Finally, we give an application to a conjecture of Kawaguchi and Silverman in the case of regular polynomial automorphisms.

2. Algebraic dynamical pairs of regular polynomial automorphism type

2.1. Definition and first properties. — Let $\mathscr{B}$ be a smooth projective complex curve and $\Lambda \subset \mathscr{B}$ a Zariski open subset. Let $\pi : \mathbb{P}^2 \times \mathscr{B} \to \mathscr{B}$ be the projection onto the first coordinate. We let $f : \mathbb{C}^2 \times \Lambda \longrightarrow \mathbb{C}^2 \times \Lambda$ be an algebraic family of regular polynomial automorphisms of $\mathbb{C}^2$. For each $\lambda \in \Lambda$, $f_\lambda(x, y) = (p_\lambda(x, y), q_\lambda(x, y))$, where p_λ, q_λ are polynomials in (x, y) that depend holomorphically on λ with $\max\deg(p_\lambda, q_\lambda) = d$ independent of λ (up to restricting Λ). We assume that the map f_λ extends as a birational map $f_\lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with

1. the only indeterminacy point I^+ of f_λ is $I^+ = [1 : 0 : 0]$,
2. the only indeterminacy point I^- of f_λ^{-1} is $I^- = [0 : 1 : 0]$.

We call such f an *algebraic family of regular polynomial automorphisms*. From our normalization, we see that $\deg p_\lambda(x, y) < \deg q_\lambda(x, y) = \deg_y q_\lambda(x, y)$ and that I^- (resp. I^+) is a super-attracting fixed point for f_λ (or f_λ^{-1}).

A classical example is given by Hénon maps:

$$f(x, y, \lambda) = (a(\lambda)y, x + p_\lambda(y), \lambda),$$

where $p : \mathbb{C} \times \Lambda \to \mathbb{C}$ is an algebraic family of degree $d > 1$ polynomials in one complex variable parametrized by the quasi-projective variety Λ with $p_\lambda(y) = p(y, \lambda)$, $a \in \mathbb{C}[\Lambda]^*$, and the support of $\mathrm{div}(a)$ is contained in the finite set $\mathscr{B} \setminus \Lambda$.