Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON THE GLOBAL DETERMINANT METHOD

Chunhui Liu

Tome 150
Fascicule 4

2022

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 699-741

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 4, tome 150, décembre 2022

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
150 (4), 2022, p. 699-741

ON THE GLOBAL DETERMINANT METHOD

by Chunhui Liu

Abstract. — In this paper, we build the global determinant method of Salberger by Arakelov geometry explicitly. As an application, we study the dependence on the degree of the number of rational points of bounded height in plane curves. We will also explain why some constants will be more explicit if we admit the generalized Riemann hypothesis.

Résumé (*Autour de la méthode globale de déterminant*). — Dans cet article, on construit la méthode globale de déterminant de Salberger par la géométrie d'Arakelov explicitement. Comme une application, on étudie la dépendance du degré du nombre de points rationnels de hauteur majorée dans courbes planes. On expliquera aussi pourquoi certaines constantes seront plus explicites si on admet l'hypothèse généralisée de Riemann.

1. Introduction

Let $X \hookrightarrow \mathbb{P}^n_K$ be a projective variety over a number field K. For every rational point $\xi \in X(K)$, we denote by $H_K(\xi)$ the height (see (1) for the

Texte reçu le 5 juillet 2021, modifié le 28 mars 2022, accepté le 18 mai 2022.

Chunhui Liu, Institute for Advanced Study in Mathematics, Harbin Institute of Technology, 150001 Harbin, P.R. China • *E-mail :* `chunhui.liu@hit.edu.cn`

Mathematical subject classification (2010). — 11D45, 11G50, 14G05, 14G25, 14G40.

Key words and phrases. — Arakelov geometry, arithmetic variety, counting rational points, determinant method, height.

Chunhui Liu was supported by Fundamental Research Funds for the Central Universities FRFCU5710010421.

doi:10.24033/bsmf.2859

definition) of ξ with respect to the above closed immersion, for example, the classic Weil height (cf. [25, §B.2, Definition]). Let

$$S(X;B)=\{\xi\in X(K)\mid H_K(\xi)\leqslant B\},$$

where $B\geqslant 1$ and the embedding morphism is omitted. By Northcott's property, the cardinality $\#S(X;B)$ is finite for a fixed $B\in\mathbb{R}$.

In order to understand the density of the rational points of X, it is an important approach to study the function $\#S(X;B)$ with the variable $B\in\mathbb{R}^+$. For different required properties of $\#S(X;B)$, numerous methods have been applied. In this article, we are interested in the uniform upper bound of $\#S(X;B)$ for all $X\hookrightarrow\mathbb{P}^n_K$ with fixed degree and dimension, and for those satisfying certain common conditions.

1.1. Determinant method. — In order to understand the function $\#S(X;B)$ of the variable $B\in\mathbb{R}^+$, we will introduce the so-called *determinant method* to study the number of rational points with bounded height in arithmetic varieties, which was proposed in [24].

1.1.1. *Basic ideas and history.* — Traditionally, the determinant method was proposed over the rational number field $\mathbb{Q}$ to avoid some extra technical troubles. In [3] (see also [35]), Bombieri and Pila proposed a method of determinant argument to study plane affine curves. The monomials of a certain degree evaluated on a family of rational points in $S(X;B)$ having the same reduction modulo some prime numbers form a matrix whose determinant is zero by a local estimate. By this method, they proved $\#S(X;B)\ll_{\delta,\epsilon}B^{2/\delta+\epsilon}$ for all $\epsilon>0$, where $\delta=\deg(X)$.

In [24], Heath-Brown generalized the method of [3] to the higher dimensional case. His idea is to focus on a subset of $S(X;B)$ whose reductions modulo a prime number are a same regular point, and he proved that this subset can be covered by a bounded degree hypersurface, which does not contain the generic point of X. Then he counted the number of regular points over finite fields and controlled the regular reductions. In [5], Broberg generalized it to the case over arbitrary number fields.

In [41, 42], Serre asked whether $\#S(X;B)\ll_X B^{\dim(X)}(\log B)^c$ is verified for all arithmetic varieties X with a particular constant c. In [24], Heath-Brown proposed a uniform version $\#S(X;B)\ll_{d,\delta,\epsilon}B^{d+\epsilon}$ for all $\epsilon>0$ with $\delta=\deg(X)$ and $d=\dim(X)$, which is called *the dimension growth conjecture*. He proved this conjecture for some special cases. Later, Browning, Heath-Brown and Salberger had some contributions to this subject, see [6, 7, 8] for the improvements of the determinant method and the proofs under certain conditions. In [39], Salberger considered the general reductions, and the multiplicities of rational points were taken into consideration, and he proved the dimension growth conjecture with certain conditions on the subvarieties of X.

1.1.2. *A global version.* — The so-called *global determinant method* was first introduced by Salberger in [40] in order to study the dimension growth conjecture mentioned above. In general, it allows one to use only one auxiliary hypersurface to cover the rational points of bounded height, and one needs to optimize the degree of this hypersurface. By the global version, he proved the dimension growth conjecture for $\deg(X)=\delta\geqslant 4$ and $\#S(X;B)\ll_\delta B^{\frac{2}{\delta}}\log B$, when X is a curve.

In [48], Walsh refined the global determinant method in [40], and he removed the $\log B$ term in [40], when X is a curve.

1.1.3. *The dependence on degree.* — Let $X\hookrightarrow\mathbb{P}^n_{\mathbb{Q}}$ be a geometrically integral variety of degree δ and dimension d. We are also interested in the dependence of the uniform upper bound of $\#S(X;B)$ on δ, in particular when X is a plane curve ($n=2$ and $d=1$).

In [47], Walkowiak studied this problem by counting integral points over $\mathbb{Z}$. In [32, Théorème 2.10], Motte obtained an estimate, which has a better dependence on B but a worse dependence on δ than that in [47, Théorème 1].

In fact, one is able to obtain a better dependence on δ by the global determinant method. In [10], Castryck, Cluckers, Dittmann and Nguyen improved [48] by giving an explicit dependence on δ. As applications, they obtained $\#S(X;B)\ll\delta^4B^{2/\delta}$, when X is a plane curve, and a better partial result of the dimension growth conjecture than that in [40], and the estimates in [47] and [32] for the case of plane curves.

In [34], the work [10] was generalized over an arbitrary global field. Before [34] was announced, Vermeulen studied the case over $\mathbb{F}_q(t)$ in [46].

Besides the uniform bounds of rational points, [10] also studied the bound of 2-torsion points of the class group of number fields, which improved the work [2, Theorem 1.1] of Bhargava, Shankar, Taniguchi, Thorne, Tsimerman and Y. Zhao.

1.1.4. *Formulation with Arakelov geometry.* — In [12, 13], H. Chen reformulated the work of Salberger [39] by Bost's slope method from Arakelov geometry developed in [4]. In this formulation, H. Chen replaced the matrix of monomials by the evaluation map, which sends a global section of a particular line bundle to its values on a family of rational points. With the slope inequalities, we can control the height of the evaluation map in the slope method, which replaces the role of Siegel's lemma in controlling heights.

There are two advantages of the Arakelov geometry approach. First, Arakelov geometry gives a natural conceptual framework for the determinant method over arbitrary number fields. Next, it is easier to obtain explicit estimates, since the constants obtained from the slope inequalities are given explicitly in general.

1.2. A global version with the formulation of Arakelov geometry. — In this article, we will construct the global determinant method over an arbitrary number field by Arakelov geometry following the strategy of [12, 13]. As a direct application, we will study the problem of counting rational points in plane curves and we consider how these upper bounds depend on the degree. Some of the ideas were inspired by [40, 48, 10].

1.2.1. *Main results.* — First, we have the control of auxiliary hypersurface below in Corollary 5.5 and Corollary 5.6, which are deduced from Theorem 5.4.

THEOREM 1.1. — *Let X be a geometrically integral hypersurface in $\mathbb{P}^n_K$ of degree δ, and*

$$a_n = \frac{n}{(n-1)\delta^{1/(n-1)}}$$

be a constant depending on n. Then there is a hypersurface of degree ϖ, which covers $S(X;B)$ but does not contain the generic point of X. In addition, we have

$$\varpi \ll_{K,n} \delta^3 B^{a_n}$$

in Corollary 5.5, and

$$\varpi \ll_{K,n} \delta^{3-1/(n-1)} B^{a_n} \max\left\{\frac{\log B}{[K:\mathbb{Q}]}, 1\right\}$$

in Corollary 5.6.

If we assume the generalized Riemann hypothesis, the above constants depending on K and n will be given explicitly in Corollary 5.5 and Corollary 5.6.

Since we apply the approach of Arakelov geometry in this article, we no longer use the technique of “change of coordinate” in [48, §3] and [10, §3.4]. Instead, we are able to obtain a uniform estimate directly.

1.2.2. *Potential applications.* — Similar to the previous applications of the determinant, the above estimates can be applied to study the uniform upper bound of the number of rational points with bounded height, where we will initiate the induction on the dimension as usual and the study of the distribution of the loci of small degree in a variety (see [40, §4] for such an example, which considered the density of conics in a cubic surface). By the above operation, we are able to obtain estimates of general arithmetic varieties from those of hypersurfaces via a suitable linear projection.

Since our method works over an arbitrary number field and gives an explicit estimate (or under some technical conditions), it is possible that further applications will work well under the same conditions and also be explicit.

As a direct application, we have the following results on counting the rational points of the bounded height in plane curves in Theorem 6.1 and Theorem 6.2.

THEOREM 1.2. — *Let X be a geometrically integral plane curve in $\mathbb{P}^2_K$ of degree δ. Then we have*

$$\#S(X;B) \ll_K \delta^4 B^{2/\delta}$$

in Theorem 6.1 and

$$\#S(X;B) \ll_K \delta^3 B^{2/\delta} \log B$$

in Theorem 6.2.

If we assume the generalized Riemann hypothesis, the above constants depending on K will be given explicitly in Theorem 6.1 and Theorem 6.2.

Theorem 6.1 generalizes [10, Theorem 2] over an arbitrary number field and gives an explicit estimate under the assumption of the generalized Riemann hypothesis. Theorem 6.2 can be viewed as a projective analogue of [10, Theorem 3] over an arbitrary number field and a better partial result of the conjecture of Heath-Brown referred to in Remark 6.3. These two estimates are better than those given in [47, Théorème 1] and [32, Théorème 2.10].

1.2.3. *The role of the generalized Riemann hypothesis.* — In this work, some explicit estimates of the distribution of prime ideals are applied. If we admit **GRH (the generalized Riemann hypothesis)** of the Dedekind zeta function of the base number field, we are able to obtain more explicit estimates; see [20], for example. Without the assumptions of GRH, it seems to be very difficult to obtain such explicit estimates over an arbitrary number field, since we do not know the zero-free region of the Dedekind zeta function. If we know the zero-free region clearly enough, for example, if we work in the rational number field $\mathbb{Q}$ or totally imaginary fields (see [44] and [22], respectively), or we just want an implicit estimate (see [37]), we do not need to suppose GRH.

1.3. Organization of the article. — This paper is organized as following. In §2, we provide some preliminaries to construct the determinant method. In §3, we formulate the global determinant method by the slope method. In §4, we give some useful estimates on the non-geometrically integral reductions, a count of multiplicities over finite fields, the distributions of some particular prime ideals, and the geometric Hilbert–Samuel function. In §5, we provide an explicit upper bound of the determinant and lower bounds of auxiliary hypersurfaces. In §6, we give two uniform upper bounds of rational points of bounded height in plane curves. In §7, under the assumption of GRH, we give some explicit estimates of the distribution of prime ideals of bounded norm in a ring of integers and explain how to apply these explicit estimates in the global determinant to get more explicit estimates. In Appendix A, we will give an explicit lower bound of a useful function induced by the local Hilbert–Samuel function.