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ANTI-GRAVITY À LA CARLOTTO-SCHOEN
[after Carlotto and Schoen]

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INTRODUCTION

In [13] Carlotto and Schoen show that gravitational fields can be used to shield gravitational fields. That is to say, one can produce spacetime regions extending to infinity where no gravitational forces are felt whatsoever, by manipulating the gravitational field around these regions. A sound-bite version of the result reads:

Theorem A (Carlotto & Schoen [13]). — *Given an asymptotically flat initial data set for vacuum Einstein equations there exist cones and asymptotically flat vacuum initial data which coincide with the original ones inside the cones and are Minkowskian outside slightly larger cones, see Figure 1.*

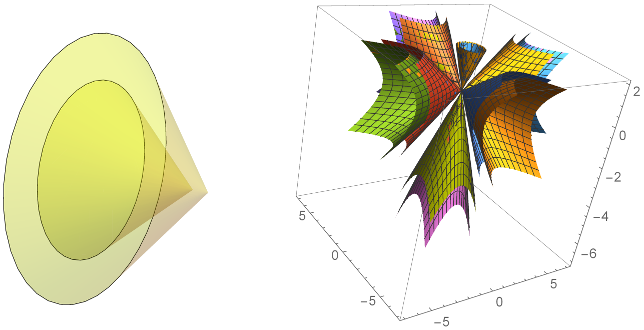

Figure 1. Left picture: The new initial data are Minkowskian outside the larger cone, and coincide with the original ones inside the smaller one. The construction can also be carried out the other way round, with Minkowskian data inside the smaller cone and the original ones outside the larger cone. Both cones extend to infinity, and their tips are located very far in the asymptotically flat region. Right picture: Iterating the construction, one can embed any finite number of distinct initial data sets into Minkowskian data, or paste-in Minkowskian data inside several cones into a given data set.

Actually, the result is true for all cones with preassigned axis and pair of apertures provided the vertex is shifted sufficiently far away in the asymptotically flat regions.

In the associated spacetimes $(\mathscr{M}, \mathbf{g})$ the metric coincides with the Minkowski metric within the domain of dependence, which we will denote by $\mathscr{D}$, of the complement of the larger cones, which forms an open subset of $\mathscr{M}$; we return to this in Section 1.6 below. Physical objects in $\mathscr{D}$ do not feel any gravitational fields. The Carlotto-Schoen gluing has effectively switched off any gravitational effects in this region. This has been achieved by manipulating vacuum gravitational fields only.

In this *séminaire* I will describe the context, define the notions used, highlight the key elements of the proof of Theorem A, and discuss some developments.

1. THE CONTEXT

1.1. Newtonian gravity

Newtonian gravity is concerned with a gravitational potential field ϕ, which solves the equation

$$\Delta \phi = 4\pi G \rho, \tag{1.1}$$

where Δ is the Laplace operator in an Euclidean $\mathbb{R}^3$ and G is Newton's constant. Up to conventions on signs, proportionality factors, and units, ρ is the matter density, which is not allowed to be negative. Isolated systems are defined by the requirement that both ρ and ϕ decay to zero as one recedes to infinity.

Freely falling bodies experience an acceleration proportional to the gradient of ϕ. So no gravitational forces exist in those regions where ϕ is constant.

Suppose that ρ has support contained in a compact set K, and that ϕ is constant on an open set Ω. Since solutions of (1.1) are analytic on $\mathbb{R}^3 \setminus K$, ϕ is constant on any connected component of $\mathbb{R}^3 \setminus K$ which meets Ω. We conclude that if Ω extends to infinity, then ϕ vanishes at all large distances. This implies, for all sufficiently large spheres $S(R)$,

$$0 = \int_{S(R)} \nabla \phi \cdot n \, d^2 S = \int_{B(R)} \Delta \phi \, d^3 V = 4\pi G \int_{B(R)} \rho \, d^3 V.$$

Since ρ is non-negative, we conclude that $\rho \equiv 0$. Equivalently, for isolated systems with compact sources, *Newtonian gravity cannot be screened away on open sets extending to large distances.*

The striking discovery of Carlotto and Schoen is, that this can be done in Einsteinian gravity.

The Newtonian argument above fails if matter with negative density is allowed. It should therefore be emphasized that the Carlotto-Schoen construction is done by manipulating vacuum initial data, without involvement of matter fields.

1.2. Einsteinian gravity, general relativistic initial data sets

Mathematical general relativity is born around 1952 with the breakthrough paper of Yvonne Choquet-Bruhat [42], showing that Einstein's field equations,

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} + \Lambda\mathbf{g}_{\mu\nu} = \frac{8\pi G}{c^4}\mathbf{T}_{\mu\nu}, \tag{1.2}$$

admit a well posed Cauchy problem. Here $\mathbf{R}_{\mu\nu}$ is the Ricci tensor of the spacetime metric $\mathbf{g}$, $\mathbf{R}$ its Ricci scalar, $\mathbf{T}_{\mu\nu}$ the energy-momentum tensor of matter fields, Λ the "cosmological constant," G is Newton's constant as before, and c is the speed of light. In vacuum $\mathbf{T}_{\mu\nu}$ vanishes, in which case (1.2) is equivalent to the requirement that $\mathbf{g}$ be Ricci-flat when moreover the vanishing of Λ is imposed. The notation $\mathbf{g}_{\mu\nu}$ indicates that the metric $\mathbf{g}$ is a two-covariant tensor field, similarly for $\mathbf{T}_{\mu\nu}$, etc.

The geometric initial data for the four-dimensional vacuum Einstein equations are a triple $(\mathscr{S}, g, K)$, where $(\mathscr{S}, g)$ is a three-dimensional Riemannian manifold and K is a symmetric two-covariant tensor field on $\mathscr{S}$. One should think of $\mathscr{S}$ as a space-like hypersurface in the vacuum Lorentzian spacetime $(\mathscr{M}, \mathbf{g})$, then g is the metric induced by $\mathbf{g}$ on $\mathscr{S}$, and K is the second fundamental form ("extrinsic curvature tensor") of $\mathscr{S}$ in $(\mathscr{M}, \mathbf{g})$.

It has already been recognized in 1927 by Darmois [37] that (g, K) are not arbitrarily specifiable, but have to satisfy a set of constraint equations,

$$R = |K|^2 - (\mathrm{tr}_g K)^2 + 2\mu + 2\Lambda, \tag{1.3}$$

$$D^i(K_{ij} - \mathrm{tr}_g K g_{ij}) = J_j, \tag{1.4}$$

where $\mu = \frac{8\pi G}{c^4}\mathbf{T}_{\mu\nu}n^\mu n^\nu$ is the matter energy density on $\mathscr{S}$ and $J_j = \frac{8\pi G}{c^4}\mathbf{T}_{\mu j}n^\mu$ the matter momentum vector, with n^μ being the unit normal to $\mathscr{S}$ in $(\mathscr{M}, \mathbf{g})$. The requirement of positivity of energy of physical matter fields translates into the *dominant energy condition*:

$$\mu \geq |J|_g, \tag{1.5}$$

where of course $|J|_g \equiv \sqrt{g(J,J)} \equiv \sqrt{g_{ij}J^iJ^j}$ (summation convention). In particular μ should be non-negative.

The constraint equations are the source of many headaches in mathematical and numerical general relativity. On the other hand, together with the energy condition, they are the source of beautiful mathematical results (1) such as the positive energy

(1) The reader is invited to consult [4, 14, 26, 35] for more details and further references.

theorems [50, 51], the Penrose inequality [10, 44], the Corvino-Schoen [32, 36] or the Carlotto-Schoen gluings.

1.3. Asymptotic flatness

Initial data for general relativistic isolated systems are typically modeled by *asymptotically flat* data with vanishing cosmological constant Λ. Actually, astrophysical observations indicate that Λ is positive. However, for the purpose of observing nearby stars, or for our stellar system, the corrections arising from Λ are negligible, they only become important at cosmological scales.

The class of asymptotically flat systems should obviously include the Schwarzschild black holes. In those, on the usual slicing by $t=$ const. hypersurfaces it holds that $K_{ij}\equiv 0$ and

$$g_{ij}=\Big(1+\frac{m}{2r}\Big)^4\delta_{ij}+O(r^{-2}), \tag{1.6}$$

in spacetime dimension four, or

$$g_{ij}=\Big(1+\frac{m}{2r^{n-2}}\Big)^{\frac{4}{n-2}}\delta_{ij}+O(r^{-(n-1)}), \tag{1.7}$$

in general spacetime dimension $n+1$. Here δ_{ij} denotes the Euclidean metric in manifestly flat coordinates. The asymptotics (1.6) is often referred to as *Schwarzschildean*, and the parameter m is called the ADM mass of the metric. Now, one can obtain initial data with non-vanishing total momentum by taking Lorentz-transformed slices in the Schwarzschild spacetime. This leads to initial data sets satisfying

$$\partial_{i_1}\cdots\partial_{i_\ell}(g_{ij}-\delta_{ij})=O(r^{-\alpha-\ell}), \tag{1.8}$$

$$\partial_{i_1}\cdots\partial_{i_k}K_{ij}=O(r^{-\alpha-k-1}), \tag{1.9}$$

with $\alpha=n-2$, for any $k,\ell\in\mathbb{N}$. Metrics g satisfying (1.8) will be called *asymptotically Euclidean.*

The flexibility of choosing $\alpha\in(0,n-2)$ in the definition of asymptotic flatness (1.8)-(1.9), as well as k,ℓ smaller than some threshold, is necessary in Theorem A. Indeed, the new initial data constructed there are *not expected* to satisfy (1.8) with $\alpha=n-2$. It would be of interest to settle the question, whether or not this is really the case.

There does not appear to be any justification for the Schwarzschildean threshold $\alpha=n-2$ other than historical. On the other hand, the threshold

$$\alpha=(n-2)/2 \tag{1.10}$$

appears naturally as the optimal threshold for a well-defined total energy-momentum of the initial data set. This has been first discussed in [15–17, 41], compare [2].

1.4. Time-symmetric initial data and the Riemannian context

Initial data are called *time-symmetric* when $K_{ij} \equiv 0$. In this case, and assuming vacuum, the *vector constraint equation* (1.4) is trivially satisfied, while the *scalar constraint equation* (1.3) becomes the requirement that $(\mathscr{S}, g)$ has constant scalar curvature R:

$$R = 2\Lambda. \tag{1.11}$$

In particular $(\mathscr{S}, g)$ should be scalar-flat when $\Lambda = 0$. (The "time-symmetric" terminology reflects the fact that a suitable reflection across $\mathscr{S}$ in the associated spacetime is an isometry.) So all statements about vacuum initial data translate immediately into statements concerning scalar-flat Riemannian manifolds. For example, the following statement is a special case of Theorem A:

THEOREM 1.1 (Carlotto & Schoen). — *Given a scalar-flat asymptotically Euclidean metric g there exist cones and scalar-flat asymptotically Euclidean metrics which coincide with g inside of the cones and are flat outside slightly larger cones.*

This theorem was one of the motivations for the proof of Theorem A. For instance, the question of existence of non-trivial, scalar-flat, asymptotically flat metrics $\hat{g}$ which are exactly flat in a half-space arises when studying complete, non-compact minimal hypersurfaces. If $\hat{g}$ is such a metric, then all hyperplanes lying in the flat half-space minimize area under compactly supported deformations which do not extend into the non-flat region. So Theorem 1.1 shows that such metrics $\hat{g}$ actually exist. This should be contrasted with the following beautiful result of Chodosh and Eichmair [12], which shows that *minimality under all compactly supported perturbations* implies flatness:

THEOREM 1.2 (Chodosh, Eichmair). — *The only asymptotically Euclidean three-dimensional manifold with non-negative scalar curvature that contains a complete non-compact embedded surface S which is a (component of the) boundary of some properly embedded full-dimensional submanifold of (M, g) and is area-minimizing under compactly supported deformations is flat $\mathbb{R}^3$, and S is a flat plane.*

The above was preceded by a related rigidity result of Carlotto [11]:

THEOREM 1.3 (Carlotto). — *Let (M, g) be a complete, three-dimensional, asymptotically Schwarzschildean Riemannian manifold with non-negative scalar curvature. If M contains a complete, properly embedded, stable minimal surface S, then (M, g) is the Euclidean space and S is a flat plane.*

Such results immediately imply non-compactness for sequences of solutions of the Plateau problem with a diverging sequence of boundaries. We note that compactness results in this spirit play a key role in the Schoen & Yau proof of the positive energy