ISSN 0012-9593

quatrième série - tome 42 fascicule 1 janvier-février 2009

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Brett KOTSCHWAR & Lei NI

Local gradient estimates of p-harmonic functions, 1/H-flow, and an entropy formula

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 42, 2009, p. 1 à 36

LOCAL GRADIENT ESTIMATES OF p-HARMONIC FUNCTIONS, $1/H$-FLOW, AND AN ENTROPY FORMULA

BY BRETT KOTSCHWAR AND LEI NI

ABSTRACT. – In the first part of this paper, we prove local interior and boundary gradient estimates for p-harmonic functions on general Riemannian manifolds. With these estimates, following the strategy in recent work of R. Moser, we prove an existence theorem for weak solutions to the level set formulation of the $1/H$ (inverse mean curvature) flow for hypersurfaces in ambient manifolds satisfying a sharp volume growth assumption. In the second part of this paper, we consider two parabolic analogues of the p-harmonic equation and prove sharp Li-Yau type gradient estimates for positive solutions to these equations on manifolds of nonnegative Ricci curvature. For one of these equations, we also prove an entropy monotonicity formula generalizing an earlier such formula of the second author for the linear heat equation. As an application of this formula, we show that a complete Riemannian manifold with nonnegative Ricci curvature and sharp L^p-logarithmic Sobolev inequality must be isometric to Euclidean space.

RÉSUMÉ. – Dans la première partie de cet article, nous établissons des estimées locales de gradient pour les fonctions p-harmoniques à l'intérieur et au bord, sur les variétés riemanniennes générales. Grâce à ces estimations et suivant une idée récente de R. Moser, nous obtenons un théorème d'existence de solutions faibles au sens de la formulation d'ensemble de niveau pour le flot $1/H$ (inverse de la courbure moyenne) des hypersurfaces dans les variétés ambiantes ayant la propriété de la croissance optimale du volume. Dans la deuxième partie, nous considérons deux types d'équations paraboliques pour les fonctions p-harmoniques et nous établissons une estimation optimale du type de Li-Yau pour les solutions positives pour ces équations sur les variétés à courbure de Ricci non-négative. Nous montrons aussi une formule de monotonie des entropies associées à ces équations. Cette formule généralise un résultat antérieur du deuxième auteur pour l'équation de la chaleur linéaire. Comme application, nous montrons que toute variété riemannienne complète à courbure de Ricci positive ou nulle et admettant une inégalité logarithmique L^p optimale est isométrique à l'espace euclidien.

The first author was supported in part by NSF grant DMS-0805834. The second author was supported in part by NSF grant DMS-0805143 and an Alfred P. Sloan Fellowship, USA.

0012-9593/01/© 2009 Société Mathématique de France. Tous droits réservés

1. Introduction

Recently, in [26], an interesting connection between the p-harmonic functions and the $1/H$ hypersurface flow (also called the inverse mean curvature flow) was established. Let v be a positive p-harmonic function, i.e., a function satisfying

$$\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0, \tag{1.1}$$

and let $u \doteqdot -(p-1)\log v$. It is easy to see that u satisfies

$$\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^p. \tag{1.2}$$

The objective in [26] is to obtain a weak solution to the $1/H$ flow in the level-set formulation of [21] on a co-compact subdomain Ω of $\mathbb{R}^n$. Where it is sufficiently regular, such a solution u will satisfy

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|, \tag{1.3}$$

that is, (1.2) with $p = 1$. Moser's strategy in [26] is to produce weak solutions to (1.3) as limits of solutions $u^{(p)}$ to (1.2) as $p \searrow 1$. He obtains the $u^{(p)}$ via the above correspondence by using explicit barriers to first solve (1.1) with appropriate Dirichlet boundary conditions.

The key ingredient in the proof of the convergence and, consequently, to the success of this strategy, is a uniform gradient estimate on the $u = u^{(p)}$. The estimate is aided by the following observation. If

$$f \doteqdot |\nabla u|^2 \doteqdot (p-1)^2\frac{|\nabla v|^2}{v^2},$$

then, expressed in terms of f, (1.2) has the equivalent form

$$f^{p/2-1}\Delta u + \left(\frac{p}{2}-1\right)f^{p/2-2}\langle \nabla f, \nabla u\rangle = f^{p/2}. \tag{1.4}$$

Using this equation, the gradient estimate in [26] follows from a boundary estimate, which in turn is derived from a C^0 estimate by way of explicit barriers, certain integral estimates on $|\nabla v|$, and a Harnack inequality for p-harmonic functions on Euclidean spaces. The construction of the barriers in particular relies on the underlying Euclidean structure of the manifold. Some precedence for both the regularization procedure and the reduction of the convergence argument to a uniform gradient estimate can be found, for example, in [14], in another context.

In the first part of this paper, we derive interior and boundary gradient estimates on a general Riemannian manifold (M, g) via the gradient estimate technique of [25, 5] and use it to establish an existence result for the $1/H$-flow on a class of complete Riemannian manifolds. A new feature of our derivation of the local estimate is a nonlinear Bochner type formula relating the nonlinear operator with its linearization.

We first start with an interior/local estimate for positive p-harmonic functions, which is of independent interest.

THEOREM 1.1. – *Assume that v is a positive p-harmonic function on the ball $B(x_0, R)$, and that on the ball $B(x_0, R)$ the sectional curvature of (M, g), $K_M \geq -K^2$. Then for any $\epsilon > 0$,*

$$\sup_{B(x_0,\frac{R}{2})}|\nabla u|^2 \leq \frac{20(n-1)}{R^2(1-\epsilon)}\left(c_{p,n} + \frac{(n-1)b^2_{p,n}}{8\epsilon}\right) + C(n,K,p,R,\epsilon) \tag{1.5}$$

where

$$
\begin{aligned}
b_{p,n} &= \left(\frac{2(p-1)}{n-1} - 2\right), \\
c_{p,n} &= \left(\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2}\right)_- + \left(\left(\frac{p}{2}+1\right)\max\{p-1,1\}\right), \\
C(n,K,p,R,\epsilon) &= \frac{(n-1)^2}{1-\epsilon}K^2 + \frac{40(n+p-2)(n-1)}{1-\epsilon}\frac{1+KR}{R^2} \\
&\quad + \frac{20\max\{p-1,1\}(n-1)}{(1-\epsilon)R^2}.
\end{aligned}
$$

Note that $b_{p,n}$, $c_{p,n}$ and $C(n,K,p,R,\epsilon)$ all stay finite as $p \to 1$. Hence Theorem 1.1 effectively gives an estimate for the gradient of the solution to $1/H$-flow. Also, if v is defined globally, by taking $R \to \infty$, then $\epsilon \to 0$, Theorem 1.1 implies that for any positive p-harmonic function v, $u = -(p-1)\log v$ satisfies

$$
|\nabla u|^2 \le (n-1)^2 K^2. \tag{1.6}
$$

The constant $(n-1)^2$ is sharp in light of the results of [24, 30] for the case $p = 2$. From the proof it is evident that when $p = 2$, one can relax the assumption $K_M \ge -K^2$ to $\mathrm{Ric} \ge -(n-1)K^2$. A direct consequence of this is that if (M,g) is a complete manifold with nonnegative sectional curvature, then any positive p-harmonic function must be a constant. In fact, using the gradient technique of this paper, one can prove that any positive p-harmonic function on a manifold with nonnegative Ricci curvature and a sectional curvature lower bound must be a constant. Indeed, this result can be obtained by only assuming that the manifold has the so-called volume doubling property and satisfies a Poincaré type inequality (see, for example, [20] as well as Section 4 for more details). Hence it holds in particular on any Riemannian manifold with nonnegative Ricci curvature. On the other hand, it is not clear whether one can obtain an estimate on $|\nabla u|^2$ such as (1.6) under the weaker assumption of a Ricci curvature lower bound.

With the help of the interior gradient estimate, by constructing suitable (local) barrier functions we can establish the following boundary estimate:

For every $\epsilon > 0$, there exists $p_0 = p(\epsilon) > 1$ such that if u satisfies equation (1.2) on Ω for some $1 < p \le p_0$, then we have the estimate

$$
|\nabla u| \le H_+ + \epsilon \tag{1.7}
$$

where H denotes the mean curvature of $\partial\Omega$ and $H_+(x) = \max\{H(x), 0\}$.

A similar boundary estimate was first proved in [21] for solutions to a different equation approximating (1.3). Our method is a modification of theirs.

With the help of the interior and boundary estimates above, and following the general scheme of [26], one can prove the existence of a *proper solution* (please see Section 4 for the definition) for a class of Riemannian manifolds which includes the asymptotically Euclidean manifolds considered in [21]. The following is a special case implied by our general existence theorem.

THEOREM 1.2. – *Let M be a complete Riemannian manifold such that its sectional curvature $K_M(x) \geq -k(r(x))$ for some nonincreasing function $k(t)$ with $\int_0^\infty tk(t)\,dt < \infty$. Let Ω be an end of M. Assume that for some $p_0 > 1$,*

$$\int_1^\infty \left(\frac{t}{V(\Omega \cap B(o,t))}\right)^{1/(p_0-1)} dt < \infty$$

and

$$\lim_{r\to\infty} \sup_{2r\leq t<\infty} \frac{t}{V(\Omega \cap B(o,t))} = 0.$$

Then (1.3) has a proper weak solution u with $\lim_{x\to\infty} |\nabla u|(x) = 0$.

The volume growth conditions in the theorem are optimal for the existence of *proper* solutions. This is shown in Section 4.

After the uniform gradient estimate, the key of the proof of the existence theorem is to construct certain bounded p-harmonic functions and obtain effective C^0-estimates of such p-harmonic functions at infinity which hold up as $p \to 1$. The estimate is somewhat delicate as we need to ensure that the limit as $p \to 1$ is a nonconstant function which tends to $+\infty$ at the spatial infinity. Here we rely crucially on an early work of Holopainen [20]. One may expect more general existence results, for example, for manifolds with a Laplacian whose L^2-spectrum has a positive lower bound [24]. However, it seems that perhaps more refined estimates may be needed.

In the second part of the paper, beginning in Section 5, we consider some nonlinear parabolic equations motivated by Theorem 1.1. First, we prove sharp gradient estimates of Li-Yau type for two nonlinear parabolic equations associated with (1.1). Both estimates proved are sharp in the case that M has nonnegative Ricci curvature and provide nonlinear generalizations of Li-Yau's estimate for the heat equation. One of these estimates, in the case $M = \mathbb{R}^n$, was obtained earlier in [11, 12] (see also [29]) in the study of the regularity of nonnegative weak solutions. We refer the readers to Section 5 and 6 for the more detailed discussions on these results.

A little surprisingly, we also obtain the following entropy formula for a class of nonlinear parabolic equations, generalizing the earlier formula for the linear heat equation in [27].

THEOREM 1.3. – *Let (M, g) be a complete Riemannian manifold. For any $p > 1$, let v be a positive solution to the equation*

$$\frac{\partial v^{p-1}}{\partial t} = (p-1)^{p-1} \operatorname{div}(|\nabla v|^{p-2}\nabla v) \tag{1.8}$$

satisfying $\int v^{p-1} d\mu = 1$. Then

$$\frac{d}{dt}\mathcal{W}_p(v,t) = -tp\int_M \left(\left|f^{p/2-1}\nabla_i\nabla_j u - \frac{1}{tp}a_{ij}\right|_A^2 + f^{p-2}R_{ij}u_iu_j\right) v^{p-1}\,d\mu$$

with $u = -(p-1)\log v$, $f = |\nabla u|^2$, $a_{ij} = g_{ij} - \frac{p-2}{p-1}\frac{v_iv_j}{|\nabla v|^2}$, $|T|_A^2 = A^{ik}A^{jl}T_{ij}T_{kl}$ for any 2-tensor T where (A^{ij}) is the inverse of (a_{ij}). The entropy

$$\mathcal{W}_p(v,t) = \int_M (t|\nabla\varphi|^p + \varphi - n)v^{p-1}\,d\mu$$