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SMALL GAPS IN THE SPECTRUM OF THE RECTANGULAR BILLIARD

BY VALENTIN BLOMER, JEAN BOURGAIN, MAKSYM RADZIWIŁŁ AND ZEÉV RUDNICK

ABSTRACT. – We study the size of the minimal gap between the first N eigenvalues of the Laplacian on a rectangular billiard having irrational squared aspect ratio α, in comparison to the corresponding quantity for a Poissonian sequence. If α is a quadratic irrationality of certain type, such as the square root of a rational number, we show that the minimal gap is roughly of size $1/N$, which is essentially consistent with Poisson statistics. We also give related results for a set of α's of full measure. However, on a fine scale we show that Poisson statistics is violated for all α. The proofs use a variety of ideas of an arithmetical nature, involving Diophantine approximation, the theory of continued fractions, and results in analytic number theory.

RÉSUMÉ. – On étudie l'écart minimal dans les N premières valeurs propres du Laplacien d'un billard rectangulaire dont le rapport des côtés est égal à $1/\sqrt{\alpha}$. On compare nos résultats avec l'écart minimal des points provenant d'une suite aléatoire poissonienne. Pour α un irrationnel quadratique d'un certain type, par exemple la racine d'un nombre rationnel, nous démontrons que l'écart minimal est approximativement de taille $1/N$. Cela est en accord avec les statistiques poissoniennes. Nous démontrons aussi un phénomène semblable pour presque tout α au sens de la mesure de Lebesgue. Cependant, à une échelle fine, de taille $1/N$, nous démontrons que l'écart minimal entre les valeurs propres et celui d'une suite poissonienne ont un comportement différent. Les démonstrations utilisent plusieurs résultats d'origine arithmétique, tels que l'approximation diophantienne, la théorie des fractions continues, et des résultats provenant de la théorie analytique des nombres.

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1. Introduction

The local statistics of the energy levels of several integrable systems are believed to follow Poisson statistics [2]. In this note we examine a variant of these statistics, the size of the minimal gap between levels, for the energy levels of a particularly simple system, a rectangular billiard. If the rectangle has width $\pi/\sqrt{\alpha}$ and height π, with aspect ratio $\sqrt{\alpha}$, then the energy levels, meaning the eigenvalues of the Dirichlet Laplacian, consist of the numbers $\alpha m^2 + n^2$ with integers $m, n \geq 1$.

The case of rational α is special: The eigenvalues lie in a lattice, in particular the nonzero gaps are bounded away from zero, and there are arbitrarily large multiplicities. We exclude this case from our discussion. If α is irrational, we get a simple spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with growth (Weyl's law)

$$\#\{j : \lambda_j \leq X\} = \#\{(m,n) : m,n \geq 1,\ \alpha m^2 + n^2 \leq X\} \sim \frac{\pi}{4\sqrt{\alpha}} X$$

as $X \to \infty$. In this setting, the pair correlation function has been shown to be Poissonian [8] for Diophantine α, see also [21] for a related problem.

We wish to study the size of the minimal gap function of the spectrum, defined as

$$\delta_{\min}^{(\alpha)}(N) = \min(\lambda_{i+1} - \lambda_i : 1 \leq i < N).$$

To set expectations, it is worth comparing with the size of the analogous quantity for some random sequences, when measured on the scale of the mean spacing between the levels in the sequence, which in our case is constant (equal to $4\sqrt{\alpha}/\pi$). For a Poissonian sequence of N uncorrelated levels with unit mean spacing, the smallest gap is almost surely of size $\approx 1/N$ [15]. In comparison, the smallest gap between the eigenphases of a random $N \times N$ unitary matrix is, on the scale of the mean spacing, almost surely of size $\approx N^{-1/3}$ [22, 1], in particular much larger than the Poisson case. The same behavior persists for the eigenvalues of random $N \times N$ Hermitian matrices (the Gaussian Unitary Ensemble) [22, 1]. For the Gaussian Orthogonal Ensemble of random symmetric matrices, is expected (though as of now not proved) that the minimal gap is of size $N^{-1/2}$. We note that the local statistics of the eigenvalues of the Laplacian for generic chaotic systems, such as non-arithmetic surfaces of negative curvature, are expected to follow the Gaussian Orthogonal Ensemble [3], while the local statistics of the zeros of the Riemann zeta function are expected to follow the Gaussian Unitary Ensemble [16, 20].

1.1. Order of growth of $\delta_{\min}^{(\alpha)}(N)$

Returning to our rectangular billiard, it is not hard to obtain lower bounds for $\delta_{\min}^{(\alpha)}(N)$, see §2.1. In the case of quadratic irrationalities, the gap function cannot shrink faster than $1/N$: for each quadratic irrationality α, there is some $c(\alpha) > 0$ so that

$$\delta_{\min}^{(\alpha)}(N) \geq \frac{c(\alpha)}{N}. \tag{1.1}$$

More generally, both for *algebraic* irrationalities and for almost every α (in the measure theoretic sense) the same argument shows

$$\delta_{\min}^{(\alpha)}(N) \gg 1/N^{1+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$, see Proposition 2.1 below. Both (1.1) and (1.2) depend on general results in diophantine approximation.

In (1.2) and elsewhere in the paper, we use Vinogradov's notation $f(N) \ll g(N)$ to mean that there are $c > 0$ and $N_0 \geq 1$ so that $|f(N)| \leq c|g(N)|$ for all $N > N_0$; and the notation $f(N) \asymp g(N)$ to mean both $f(N) \ll g(N)$ and $g(N) \ll f(N)$. Implied constants may always depend on α and ε where applicable.

Much more work needs to be done to obtain good upper bounds for $\delta_{\min}^{(\alpha)}(N)$, i.e., to explicitly construct small gaps.

We show in Proposition 2.2 below that for any irrational α, we have

$$\delta_{\min}^{(\alpha)}(N) \ll N^{-1/2} \tag{1.3}$$

for all N. By the same argument, we can also display α where $\delta_{\min}^{(\alpha)}(N) \ll N^{-A}$ for any $A > 0$ by taking α to be suitable Liouville numbers. However these form a measure zero set and are atypical.

For certain quadratic irrationalities we show that the minimal gap can be almost as small as $1/N$:

THEOREM 1.1. – *If the squared aspect ratio is a quadratic irrationality of the form* $\alpha = \sqrt{r}$, *with r rational, then*

$$\delta_{\min}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}}$$

for every $\varepsilon > 0$ *and all* N.

We can also deal with other quadratic irrationalities, such as the golden mean. We refer to Section 6 for more general results. In particular, we show in this section that there exist quadratic irrationalities α such that the stronger result

$$\delta_{\min}^{(\alpha)}(N) \ll 1/N \tag{1.4}$$

holds for all N. An explicit example is the square of the golden mean $\alpha = (3 + \sqrt{5})/2$.

Moving away from quadratic irrationalities, where our results are deterministic, we turn to generic in measure α.

THEOREM 1.2. – *For almost all* $\alpha > 0$ *(in the sense of Lebesgue measure) we have*

$$\delta_{\min}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}} \tag{1.5}$$

for any $\varepsilon > 0$ *and all* N.

We summarize the preceding results by stating that the order of growth of $\delta_{\min}^{(\alpha)}(N) \approx 1/N$ is consistent with Poisson statistics for certain special and also generic in measure α. However, as we now explain, finer details of Poisson statistics are always violated.