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COUNTING SPECIAL LAGRANGIAN FIBRATIONS IN TWISTOR FAMILIES OF K3 SURFACES

BY Simion FILIP
Appendix by Nicolas Bergeron and Carlos Matheus

Abstract. – The number of closed billiard trajectories in a rational-angled polygon grows quadratically in the length. This paper gives an analogue on K3 surfaces, by considering special Lagrangian tori. The analogue of the angle of a billiard trajectory is a point on a twistor sphere, and the number of directions admitting a special Lagrangian torus fibration with volume bounded by V grows like V^{20} with a power-saving term. Bergeron-Matheus have explicitly estimated the exponent of the error term as $20 - \frac{4}{697633}$. The counting result on K3 surfaces is deduced from a count of primitive isotropic vectors in indefinite lattices, which is in turn deduced from equidistribution results in homogeneous dynamics.

Résumé. – Le nombre de trajectoires fermées des billards dans un polygone à angles rationnels a une croissance quadratique comme fonction de la longueur. Cet article donne un analogue sur les surfaces K3, en considérant des tores Lagrangiens spéciaux. L'analogue de l'angle d'une trajectoire de billard est un point sur une sphère de twisteur, et le nombre de directions admettant une fibration en lagrangiens spéciaux avec un volume borné par V croit comme V^{20} avec un terme d'erreur. Bergeron-Matheus ont explicitement estimé l'exposant du terme d'erreur à $20 - \frac{4}{697633}$. Le comptage sur les surfaces K3 est déduit à partir d'un comptage de vecteurs isotropes primitifs dans des réseaux indéfinis, qui est à son tour déduit des résultats d'équidistribution en dynamique homogène.

1. Introduction

1.1. Motivations

Billiards. – Consider a regular n-gon and billiard trajectories in it. Veech [29] proved that the number of *closed* billiard trajectories of length at most L is asymptotic to $c_n L^2$ for an explicit constant c_n. For general rational-angled polygons Masur [24] proved that the number of closed trajectories has quadratic upper and lower bounds, and results of Eskin, Mirzakhani, and Mohammadi [11] imply a quadratic asymptotic in an averaged sense. For a general polygon, it is not known if a single closed trajectory exists.

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Translation surfaces. – These results are proved by studying the moduli space of "flat" or "translation" surfaces, i.e., Riemann surfaces X equipped with a holomorphic 1-form Ω. These carry a flat metric $\frac{\sqrt{-1}}{2}\Omega\wedge\overline{\Omega}$ with singularities at the zeros of Ω. To go from a rational-angled polygon to a surface, the polygon is unfolded by reflections in the sides to finitely many copies in $\mathbb{R}^2 \cong \mathbb{C}$ and the sides are glued by translations, so that the 1-form $\Omega := dz$ on $\mathbb{C}$ descends to the glued surface.

Straight lines. – Of course, closed geodesics are just length-minimizing curves. But on a translation surface (X, Ω) a horizontal curve can also be described as one for which the imaginary part $\mathrm{Im}\Omega$ restricts to zero. Similarly, straight lines at angle θ correspond to those for which $\mathrm{Im}\left(e^{\sqrt{-1}\theta}\Omega\right)$ restricts to zero.

Another feature of closed geodesics on a translation surface is that they occur in families: a small parallel deformation will again close up. Moreover, closed geodesics occur in a dense (on the unit circle) set of directions.

K3 surfaces. – We consider an extension of the above constructions to K3 surfaces—compact complex 2-dimensional manifolds which admit a nowhere vanishing holomorphic 2-form Ω and are simply connected. By Yau's solution of the Calabi conjecture [30] on a K3 any Kähler cohomology class has a unique Ricci-flat Kähler representative ω.

Special Lagrangians. – One generalization of straight lines on a translation surface to higher dimensions are special Lagrangian manifolds. Namely, on a complex n-manifold X with a holomorphic n-form Ω and Ricci-flat Kähler metric ω, a submanifold $L \subset X$ is *special Lagrangian* if:

(i) L is Lagrangian for the symplectic form ω (i.e., $\dim_{\mathbb{R}} L = n$ and $\omega|_L \equiv 0$).

(ii) The restriction $\Omega|_L$ gives the Riemannian volume form on L (in particular $\mathrm{Im}\Omega|_L \equiv 0$).

Note that special Lagrangians are automatically minimal surfaces, i.e., locally volume minimizing (because they are calibrated manifolds).

Fibrations. – We'll regard special Lagrangian tori as an analogue on K3 surfaces of closed billiard trajectories. When a single such torus exists on a K3 surface X, it deforms to give a special Lagrangian fibration $X \to B$ with finitely many singular fibers. Fibrations $X \to B$ will always be assumed to have connected fibers and two fibrations will be regarded as equivalent if there is a map on the bases $B \to B'$ such that the corresponding diagram commutes.

Twistor families. – A Ricci-flat K3 surface (X, Ω, ω) naturally sits in a family $\mathscr{X} \to \mathbb{S}^2$ in which all the fibers are isometric as Riemannian manifolds, but the complex structure changes. The points on the base $\mathbb{S}^2$ of such a twistor family will serve as directions in which we will look for special Lagrangian fibrations.

Special Lagrangian ↔ holomorphic correspondence. – If a fiber $(\mathcal{X}_{t_0}, \Omega_{t_0}, \omega_{t_0})$ of a twistor family admits a special Lagrangian fibration for some $t_0 \in \mathbb{S}^2$, then the same fibration is special Lagrangian for t in an entire equator containing t_0. At the poles corresponding to that equator, the same special Lagrangian fibration becomes an *elliptic* fibration, i.e., the fibers are complex elliptic curves for the corresponding complex structure (see Proposition 2.3.3). In both types of fibrations the Riemannian volumes of each fiber agree and give a notion of volume of a fibration.

1.2. Results

The geometry of a K3 surface is directly related to the Hodge structure on its middle cohomology group. Recall that the integral cohomology of a K3 surface equipped with cup product is isomorphic to the even unimodular lattice $\mathbb{I}_{3,19}$ of signature (3, 19). This lattice is unique up to isomorphism and will be denoted by $\Lambda_{\mathbb{Z}}$ and extensions of scalars by a ring R will be denoted Λ_R.

The information required to define a twistor family is encoded in a positive-definite 3-plane $P \subset \Lambda_{\mathbb{R}}$ which defines a semi-norm

$$\|e\|_P := \sup_{\kappa \in P, \kappa^2 = 1} \kappa \cdot e.$$

Moreover, using the decomposition $\Lambda_{\mathbb{R}} = P \oplus P^{\perp}$ we get a decomposition $e = e_P \oplus e_{P^\perp}$ and we clearly have $e_P \cdot e_P = \|e\|_P^2$.

In order to perform a count of special Lagrangian fibrations we first establish the following counting result in the homogeneous setting (see Theorem 3.1.3 for a precise statement).

Theorem A. – *The number $N(V)$ of primitive integral isotropic*[(1)] *vectors e with $\|e\|_P \le V$ satisfies:*

$$N(V) = C \cdot V^{20} + O(V^{20-\delta})$$

for a universal constant C, independent of the plane P.

Moreover, the unit vectors $\frac{1}{\|e\|_P} e_P$ quantitatively equidistribute on the unit sphere of P.

[Bergeron-Matheus, appendix] *The above asymptotic holds for any constant δ less than $\frac{4}{697633} \approx 6 \cdot 10^{-6}$.*

The leading constant $C > 0$ that appears in the theorem is the ratio of volumes of two homogeneous spaces. It is computed in § 3.1.6 up to rational factors to be $C \in (\pi^{20}\zeta(11))^{-1} \cdot \mathbb{Q}$.

The following counting result for K3 surfaces then follows by combining classical results on the geometry of K3 surfaces with Theorem A.

Theorem B. – *Let $\mathcal{X} \to \mathbb{S}^2$ be a generic (in the sense of Definition 2.2.7) twistor family of K3 surfaces. Let $N(V)$ be the number of elliptic fibrations of volume at most V which occur in some member of the twistor family. Then we have the asymptotic*

$$N(V) = C \cdot V^{20} + O(V^{20-\delta})$$

(1) That is: $e \in \Lambda_{\mathbb{Z}}$, $e^2 = 0$, and $e \neq k \cdot e'$ with $k \in \mathbb{N}$, $e' \in \Lambda_{\mathbb{Z}}$