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# TAUTOLOGICAL CLASSES WITH TWISTED COEFFICIENTS

BY DAN PETERSEN, MEHDI TAVAKOL AND QIZHENG YIN

ABSTRACT. – Let $M_g$ be the moduli space of smooth genus $g$ curves. We define a notion of Chow groups of $M_g$ with coefficients in a representation of $\mathrm{Sp}(2g)$, and we define a subgroup of tautological classes in these Chow groups with twisted coefficients. Studying the tautological groups of $M_g$ with twisted coefficients is equivalent to studying the tautological rings of all fibered powers $C_g^n$ of the universal curve $C_g \to M_g$ simultaneously. By taking the direct sum over all irreducible representations of the symplectic group in fixed genus, one obtains the structure of a twisted commutative algebra on the tautological classes. We obtain some structural results for this twisted commutative algebra, and we are able to calculate it explicitly when $g \leq 4$. Thus we completely determine the tautological rings of all fibered powers of the universal curve over $M_g$ in these genera. We also give some applications to the Faber conjecture.

RÉSUMÉ. – Notons par $M_g$ l'espace de modules des courbes lisses de genre $g$. Nous définissons une notion de groupes de Chow de $M_g$ à coefficients dans une représentation de $\mathrm{Sp}(2g)$, et nous définissons en outre un sous-groupe de classes tautologiques dans ces groupes de Chow à coefficients tordus. L'étude des groupes tautologiques de $M_g$ à coefficients tordus est équivalente à l'étude simultanée des anneaux tautologiques de toutes les puissances fibrées $C_g^n$ de la courbe universelle $C_g \to M_g$. En prenant la somme directe de toutes les représentations irréductibles du groupe symplectique en genre fixe, on obtient sur les classes tautologiques la structure d'une algèbre tordue commutative. Nous obtenons des résultats structurels pour cette algèbre tordue commutative, et nous la calculons explicitement lorsque $g \leq 4$. Ainsi, nous déterminons complètement les anneaux tautologiques de toutes les puissances fibrées de la courbe universelle sur $M_g$ pour $g \leq 4$. Quelques applications à la conjecture de Faber sont données.

## 1. Introduction

Say that $g \geq 2$, and let $C_g^n$ be the moduli space of smooth genus $g$ curves with $n$ ordered not necessarily distinct marked points. Equivalently, $C_g^n$ is the $n$-fold fibered power of the universal curve over $M_g$ with itself. Suppose that we want to study the cohomology of the spaces $C_g^n$. A natural approach is to apply the Leray-Serre spectral sequence for the fibration


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$f: C_g^n \to M_g$ that forgets the $n$ markings. Since $f$ is smooth and proper, the spectral sequence degenerates by Deligne's decomposition theorem [8], and

$$H^k(C_g^n, \mathbf{Q}) \cong \bigoplus_{p+q=k} H^p(M_g, R^q f_* \mathbf{Q}).$$

To each dominant weight $\lambda$ of $\mathrm{Sp}(2g)$ there is associated a local system $\mathbb{V}_{\langle\lambda\rangle}$ on $A_g$, the moduli space of principally polarized abelian varieties of dimension $g$. The sheaves $R^q f_* \mathbf{Q}$ decompose into direct sums of local systems $\mathbb{V}_{\langle\lambda\rangle}$, where we use the notation $\mathbb{V}_{\langle\lambda\rangle}$ also for their pullback along the Torelli map. The local systems $\mathbb{V}_{\langle\lambda\rangle}$ occuring as summands of $Rf_*\mathbf{Q}$ are precisely those with $|\lambda| \leq n$.

It follows that the two collections of cohomology groups

$$H^\bullet(C_g^n, \mathbf{Q}) \text{ for } n \leq N \qquad \text{and} \qquad H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \text{ for } |\lambda| \leq N$$

contain more or less the same information. However, this information is "packaged" in a much more efficient way in the local systems. The cohomology groups of $C_g^n$ are generally very large, but when expressed in terms of local systems we see that most of the cohomology just encodes how the complex $Rf_*\mathbf{Q}$ decomposes into summands—that is, it encodes the Künneth formula for the $n$-fold self-product of a genus $g$ curve, and some representation theory of $\mathrm{Sp}(2g)$. By studying the local systems we may focus our attention on the "interesting" part of the cohomology in a systematic way.

Our first goal of this paper is to do the same thing for the *tautological rings* of $C_g^n$. We remind the reader that the tautological ring $R^\bullet(C_g^n)$ is the subalgebra of $\mathrm{CH}^\bullet(C_g^n)$ generated by the classes of the diagonal loci $\Delta_{ij}$ where two markings coincide, the classes $\psi_1, \dots, \psi_n$ which are the Chern classes of the $n$ cotangent line bundles at the marked points, and the Morita-Mumford-Miller classes $\kappa_d$. The image of $R^\bullet(C_g^n)$ in cohomology under the cycle class map is denoted $RH^\bullet(C_g^n)$.

We will be able to define tautological cohomology groups $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \subseteq H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$, with the property that the collections of tautological groups

$$RH^\bullet(C_g^n) \text{ for } n \leq N \qquad \text{and} \qquad RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \text{ for } |\lambda| \leq N$$

bear exactly the same relation to each other as the collections of cohomology groups $H^\bullet(C_g^n, \mathbf{Q})$ and $H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$. Thus we are able to decompose the tautological groups of $C_g^n$ into pieces indexed by local systems; the tautological groups of the local systems package all the information about the tautological groups of $C_g^n$ in a much more efficient way, and working with twisted coefficients allows us to "zoom in" on particularly interesting parts of the tautological groups. Moreover, the groups $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$ turn out to be more computable than the groups $RH^\bullet(C_g^n)$.

In fact, we will actually not only do this on the level of cohomology groups, but for Chow groups. (The results are new already on the level of cohomology, though.) For this we should not work with local systems on $M_g$, but with relative Chow motives over the base $M_g$. Instead of decomposing the complex $Rf_*\mathbf{Q}$ into local systems $\mathbb{V}_{\langle\lambda\rangle}$, we will decompose the Chow motive $h(C_g^n/M_g)$ into Chow motives $\mathbf{V}_{\langle\lambda\rangle}$ which are motivic lifts of the local systems $\mathbb{V}_{\langle\lambda\rangle}$. Once the correct framework is in place, working with motives rather than local systems provides no extra difficulties.

The utility of working with the local systems is illustrated by our Theorem 10.1, in which we completely determine all tautological groups with twisted coefficients when $g = 2, 3, 4$. It is an easy matter to compute from Theorem 10.1 the ranks of all the groups $R^k(C_g^n)$ when $g \leq 4$, the decompositions of these tautological groups into $\mathfrak{S}_n$-representations, and the socle pairing. Thus a lot of useful information about the tautological rings is encoded in a few lines of information about the local systems.

Since the tautological rings are defined in terms of explicit generators, understanding the tautological rings is equivalent to finding the complete list of relations between these generators. A conjectural complete description of the tautological rings was formulated by Faber [12]. Namely, a theorem of Looijenga [43] asserts that $R^{g-2+n}(C_g^n) \cong \mathbf{Q}$, and that the tautological ring vanishes above this degree. Thus any two monomials of degree $g - 2 + n$ in the generators of the tautological ring are proportional to each other, and the proof of the $\lambda_g\lambda_{g-1}$-conjecture [19, 21] gives explicit proportionalities. (In fact, both Looijenga's theorem and the proportionalities were part of Faber's original conjecture.) What Faber then conjectured was that any possible relation which is consistent with the pairing into the top degree is a true relation; that is, the ring $R^\bullet(C_g^n)$ should satisfy Poincaré duality. The general belief now is that this conjecture should fail. One reason is that the original conjecture was later extended to a "trinity" of conjectures for the spaces $M_{g,n}^{\mathrm{rt}}$, $M_{g,n}^{\mathrm{ct}}$ and $\overline{M}_{g,n}$ [56, 13], and the conjectures for $\overline{M}_{2,n}$ and $M_{2,n}^{\mathrm{ct}}$ are known to fail when $n \geq 20$ and $n \geq 8$, respectively [62, 60]. The Faber conjecture for the spaces $M_{g,n}^{\mathrm{rt}}$ is equivalent to the Faber conjecture for $C_g^n$ [59], and is still open. It has more recently been conjectured that Pixton's extension of the FZ relations (see Section 9) gives rise to all relations between tautological classes, and this conjecture is known to contradict the Faber conjecture [63].

An interesting aspect of our work is that even though the decomposition of the tautological groups $R^\bullet(C_g^n)$ into pieces indexed by representations of $\mathrm{Sp}(2g)$ is not compatible with the ring structure, the multiplication into the top degree behaves very well: the matrix describing the top degree pairing is block diagonal with respect to our decomposition of the tautological groups. This has the consequence that the Faber conjecture can be fruitfully studied from the perspective of the motives $\mathbf{V}_{\langle\lambda\rangle}$—Poincaré duality can be checked for each $\mathbf{V}_{\langle\lambda\rangle}$ separately. Using this we show that the Faber conjecture is true for the moduli space $C_g^n$ (hence also the space $M_{g,n}^{\mathrm{rt}}$) when $g \leq 4$ and $n$ is arbitrary, and we make some progress in trying to understand likely failures of the Faber conjectures in higher genera.

A completely different perspective on our results is provided by work of Kawazumi-Morita and Hain. For a fixed genus $g \geq 2$, one can define a structure of commutative ring on the direct sum

$$\mathsf{T}_g = \bigoplus_\lambda H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \mathbb{V}_{\langle\lambda\rangle}^*,$$

where the direct sum is taken over all dominant weights $\lambda$ of $\mathrm{Sp}(2g)$. Let $\mathsf{A}_g = \bigwedge \mathbb{V}_{\langle 1,1,1\rangle}^*/(\mathbb{V}_{\langle 2,2\rangle}^*)$ denote the exterior algebra on the representation $\mathbb{V}_{\langle 1,1,1\rangle}^*$, modulo the ideal generated by the subrepresentation $\mathbb{V}_{\langle 2,2\rangle}^* \subset \bigwedge^2 \mathbb{V}_{\langle 1,1,1\rangle}^*$. If $\mathbb{V}_{\langle 1,1,1\rangle}^*$ is placed in degree 1, then one can define a natural $\mathrm{Sp}(2g)$-equivariant homomorphism of graded commutative rings $\varphi\colon \mathsf{A}_g \to \mathsf{T}_g$. In particular, we get a morphism between the subalgebras of symplectic invariants,

$$\varphi^{\mathrm{Sp}(2g)}\colon \mathsf{A}_g^{\mathrm{Sp}(2g)} \to \mathsf{T}_g^{\mathrm{Sp}(2g)} = H^\bullet(M_g, \mathbf{Q}).$$