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THE PLURIPOTENTIAL CAUCHY-DIRICHLET PROBLEM FOR COMPLEX MONGE-AMPÈRE FLOWS

BY Vincent GUEDJ, Chinh H. LU AND Ahmed ZERIAHI

ABSTRACT. – We develop the first steps of a parabolic pluripotential theory in bounded strongly pseudo-convex domains of $\mathbb{C}^n$. We study certain degenerate parabolic complex Monge-Ampère equations, modeled on the Kähler-Ricci flow evolving on complex algebraic varieties with Kawamata log-terminal singularities. Under natural assumptions on the Cauchy-Dirichlet boundary data, we show that the envelope of pluripotential subsolutions is semi-concave in time and continuous in space, and provides the unique pluripotential solution with such regularity.

RÉSUMÉ. – Nous développons une théorie pluripotentielle parabolique sur un domaine strictement pseudo-convexe borné de $\mathbb{C}^n$. Nous étudions certaines équations de Monge-Ampère complexes paraboliques dégénérées, modelées sur le flot de Kähler-Ricci sur les variétés algébriques complexes à singularités Kawamata log-terminales. Sous des hypothèses naturelles sur les données de Cauchy-Dirichlet, nous montrons que l'enveloppe des sous-solutions pluripotentielles est semi-concave en temps et continue en espace, et qu'elle est l'unique solution pluripotentielle avec une telle régularité.

Introduction

The Ricci flow, first introduced by Hamilton [18] is the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature. If the Ricci flow starts from a Kähler metric, the evolving metrics remain Kähler and the resulting PDE is called the Kähler-Ricci flow.

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It is expected that the Kähler-Ricci flow can be used to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time. Solving the Kähler-Ricci flow boils down to solving a parabolic scalar equation modeled on

$$\det\left(\frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_k}(t,z)\right) = e^{\partial_t u_t(z)+H(t,z)+\lambda u_t(z)}$$

where $t \mapsto u_t(z) = u(t,z)$ is a smooth family of strictly plurisubharmonic functions in $\mathbb{C}^n$, $\lambda \in \mathbb{R}$ and $g = e^H$ is a smooth and positive density.

It is important for geometric applications to study *degenerate* versions of these complex Monge-Ampère flows, where the functions u_t are no longer smooth nor strictly plurisubharmonic, and the densities may vanish or blow up (see [30, 4, 29, 11] and the references therein).

A viscosity approach has been developed recently in [10], following its elliptic counterpart [9, 19, 20]. While the viscosity theory is very robust, it requires the data to be continuous hence has a limited scope of applications. Several geometric situations encountered in the Minimal Model program (MMP) necessitate one to deal with Kawamata log-terminal (klt) singularities. The viscosity approach breaks down in these cases and a more flexible method is necessary.

There is a well established pluripotential theory of weak solutions to degenerate elliptic complex Monge-Ampère equations, following the pioneering work of Bedford-Taylor [1, 2]. This theory allows to deal with L^p-densities as established in a corner stone result of Kołodziej [25], which provides a great generalization of [32].

No similar theory has ever been developed on the parabolic side. The purpose of this article, the first of a series on this subject, is to develop a pluripotential theory for degenerate complex Monge-Ampère flows. This article settles the foundational material for this theory and focuses on solving the Cauchy-Dirichlet problem in domains of $\mathbb{C}^n$.

We consider the following family of Monge-Ampère flows

$$dt \wedge (dd^c u)^n = e^{\partial_t u + F(t,z,u)} g(z) dt \wedge dV, \qquad \text{(CMAF)}$$

in $\Omega_T :=]0,T[\times \Omega$, where dV is the Euclidean volume form on $\mathbb{C}^n$ and

— $T > 0$ and $\Omega \Subset \mathbb{C}^n$ is a bounded strictly pseudoconvex domain;

— $F(t,z,r)$ is continuous in $[0,T[\times \Omega \times \mathbb{R}$, increasing in r, bounded in $[0,T[\times \Omega \times J$, for each $J \Subset \mathbb{R}$;

— $(t,r) \mapsto F(t,\cdot,r)$ is uniformly Lipschitz and semi-convex in (t,r);

— $g \in L^p(\Omega)$, $p > 1$, and $g > 0$ almost everywhere ;

— $u : [0,T[\times \Omega \to \mathbb{R}$ is the unknown function.

Here $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)/2$ so that $dd^c = i\partial\overline{\partial}$ and $(dd^c u)^n$ represents the determinant of the complex Hessian of u in space (the complex Monge-Ampère operator) whenever u is $\mathcal{C}^2$-smooth.

For less regular functions u, the Equation (CMAF) should be understood in the weak sense of pluripotential theory as we explain in Section 2.

We let $\mathcal{P}(\Omega_T)$ denote the set of *parabolic potentials*, i.e., those functions $u : \Omega_T \to [-\infty, +\infty[$ defined in $\Omega_T =]0,T[\times \Omega$ and satisfying the following conditions:

— for any $t \in]0,T[$, $u(t,\cdot)$ is plurisubharmonic in Ω;

— the family $\{u(\cdot,z)\,;\ z \in \Omega\}$ is locally uniformly Lipschitz in $]0,T[$.

We study in Section 1 basic properties of parabolic potentials. We show in Lemma 1.6 that if $u \in \mathcal{P}(\Omega_T)$ and is bounded from above in Ω_T then it can be uniquely extended as an upper-semicontinuous function in $[0,T[\times\Omega$ such that $u(0,\cdot)$ is plurisubharmonic in Ω. We show that parabolic potentials satisfy approximate submean-value inequalities (Lemma 1.8) and enjoy good compactness properties (Proposition 1.17).

We show in Section 2 that parabolic complex Monge-Ampère operators are well defined on $\mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$ and enjoy nice continuity properties, allowing to make sense of pluripotential sub/super/solutions to (CMAF) (see Definition 3.1). A crucial convergence property is obtained in Proposition 2.9, under a semi-concavity assumption on the family of parabolic potentials.

A *Cauchy-Dirichlet boundary data* is a function h defined on the parabolic boundary of Ω_T denoted by

$$\partial_0\Omega_T := ([0,T[\times\partial\Omega) \cup (\{0\}\times\Omega),$$

such that

— the restriction of h on $[0,T[\times\partial\Omega$ is continuous;

— the family $\{h(\cdot,z)\,;\ z \in \partial\Omega\}$ is locally uniformly Lipschitz in $]0,T[$;

— h satisfies the following compatibility condition : $\forall\zeta \in \partial\Omega$,

$$h_0 := h(0,\cdot) \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) \text{ and } \lim_{\Omega\ni z\to\zeta} h(0,z) = h(0,\zeta). \tag{0.1}$$

The Cauchy-Dirichlet problem for the parabolic Equation (CMAF) with Cauchy-Dirichlet boundary data h consists in finding $u \in \mathcal{P}(\Omega_T) \cap L^\infty(\Omega_T)$ such that (CMAF) holds in the pluripotential sense in Ω_T and the following Cauchy-Dirichlet boundary conditions are satisfied :

$$\forall(\tau,\zeta) \in [0,T[\times\partial\Omega,\quad \lim_{\Omega_T\ni(t,z)\to(\tau,\zeta)} u(t,z) = h(\tau,\zeta), \tag{0.2}$$

$$\lim_{t\to 0^+} u_t = h_0 \quad \text{in} \quad L^1(\Omega). \tag{0.3}$$

In this case we say that u *is a solution to the Cauchy-Dirichlet problem for the Equation* (CMAF) *with boundary values h.*

Observe that a solution u to the Equation (CMAF) has plurisubharmonic slices in Ω and the Cauchy condition (0.3) implies by a classical result in pluripotential theory that $(\limsup_{t\to 0} u_t)^* = h_0^* \in \mathrm{PSH}(\Omega)$, hence $h_0 = h_0^* \in \mathrm{PSH}(\Omega)$. This observation shows that the Cauchy data h_0 must be plurisubharmonic as it is required in the compatibility condition (0.1).

For a solution to the Cauchy-Dirichlet problem for the Equation (CMAF), the Cauchy condition (0.3) implies that

$$\forall z \in \Omega,\quad \lim_{t\to 0^+} u_t(z) = h_0(z).$$

It is possible to consider less regular initial Cauchy data $h(0,\cdot)$ (see [28, 27]), but we will not pursue this here.