ISSN 0012-9593

quatrième série - tome 56 fascicule 1 janvier-février 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 56, 2023, p. 127 à 196

THE UNRUH STATE FOR MASSLESS FERMIONS ON KERR SPACETIME AND ITS HADAMARD PROPERTY

BY Christian GÉRARD, Dietrich HÄFNER
AND Michał WROCHNA

Abstract. – We give a rigorous definition of the Unruh state in the setting of massless Dirac fields on slowly rotating Kerr spacetimes. In the black hole exterior region, we show that it is asymptotically thermal at Hawking temperature on the past event horizon. Furthermore, we demonstrate that in the union of the exterior and interior regions, the Unruh state is pure and Hadamard. The main ingredients are the Häfner-Nicolas scattering theory, new microlocal estimates for characteristic Cauchy problems and criteria on the level of square-integrable solutions.

Résumé. – Nous donnons une définition rigoureuse de l'état de Unruh dans le cadre de champs de Dirac quantiques sans masse sur des espaces-temps de Kerr en rotation lente. Dans la région extérieure du trou noir, nous montrons que l'état de Unruh est asymptotiquement thermal à la température de Hawking sur l'horizon des événements passés. De plus, nous démontrons que dans l'union des régions extérieure et intérieure, l'état de Unruh est un état pur et qu'il vérifie la condition de Hadamard. Les principaux ingrédients sont la théorie de diffusion de Häfner-Nicolas, de nouvelles estimations microlocales pour les problèmes de Cauchy caractéristiques ainsi qu'un critère pour la propriété de Hadamard formulé en termes de solutions de carré intégrable.

1. Introduction and summary

1.1. Introduction

One of the major open problems in mathematical Quantum Field Theory on curved spacetimes is to determine the final quantum state arising from the collapse into a black hole and to describe its thermodynamical properties.

In the last decade, valuable insight has been gained especially from simplified models in which the black hole is eternal and non-rotating.

Most notably, years after Unruh's proposal for a distinguished state on Schwarzschild spacetime [58] and subsequent developments, including e.g., works by Candelas [4] and Dimock-Kay [18, 17], a rigorous definition was eventually provided by Dappiaggi-Moretti-Pinamonti [12]. The same authors gave also more clue to the physical relevance of the Unruh

doi:10.24033/asens.2530

state by proving that it satisfies the *Hadamard condition* on the union of the exterior and the interior region, thus ruling out infinite accumulation of energy at the event horizon. The remarkable fact is that imposing the Hadamard condition singles out a state at the *Hawking temperature* at the event horizon.

The Hadamard condition was also shown recently for the Unruh state on Reissner-Nordström–de Sitter spacetime by Hollands-Wald-Zahn [41], who used it as a reference state to demonstrate the quantum instability of the Cauchy horizon in that setting.

The essential feature of e.g., Schwarzschild spacetime that makes it more tractable than the case of rotating black holes is the existence of a Killing vector field which is time-like in the whole exterior region. Schwarzschild spacetime has also the special structure of a *static bifurcate Killing horizon*, which makes it also possible to consider a different distinguished state, the *Hartle-Hawking-Israel* state [36, 42], conjectured in the '70s to be well-defined on the whole Kruskal-Szekeres extension. While its uniqueness is known since the work of Kay-Wald [43], its rigorous construction and the proof of its Hadamard property were established relatively recently by Sanders [57], followed by a generalization to the stationary case by Gérard [25]. Although believed to be too idealized to describe the final black hole collapse state accurately, the Hartle-Hawking-Israel state is an important theoretical model nevertheless in view of its high level of symmetry and its connection to black hole thermodynamics [61].

In the physically more realistic rotating case, however, there is no rigorous result so far that gives the existence of a distinguished Hadamard state. The absence of a global time-like Killing vector field causes severe difficulties both on the conceptual and technical level, which are expressed in the following non-existence theorems:

1. The Kay-Wald theorem asserts the non-existence of a state which is invariant under the flow of the Killing field $v_{\mathscr{H}}$ that generates the horizon, under the assumption that a certain *superradiance* property holds true [43]. The latter is conjectured to be verified in the case of bosonic fields. (Note that there is no superradiance for fermions in this sense.)

2. A theorem due to Pinamonti-Sanders-Verch asserts that a thermal state associated with a complete Killing vector field v cannot be Hadamard if there is a point at which v is space-like [53]. This result holds true in a broad setting including bosonic and fermionic non-interacting fields.

Focusing our attention on the exterior Kerr spacetime $(\mathrm{M_I}, g)$ for the moment, the closest analogues of the time-like Killing vector field ∂_t on Schwarzschild are the two Killing vector fields $v_{\mathscr{H}} = \partial_t + \Omega_{\mathscr{H}}\partial_\varphi$ (the generator of the past horizon $\mathscr{H}_-$; the constant $\Omega_{\mathscr{H}}$ is the angular velocity of $\mathscr{H}_-$) and $v_{\mathscr{I}} = \partial_t$ (the generator of past null infinity $\mathscr{I}_-$), each of which is time-like only in a subregion of the exterior. While the result (2) (and in all likelihood (1) as well in the bosonic case) implies bad properties of any state in the exterior region that is thermal with respect to either $v_{\mathscr{H}}$ or $v_{\mathscr{I}}$, we propose instead to split the solution space into two parts, and use $v_{\mathscr{H}}$ for solutions coming from $\mathscr{H}_-$ and $v_{\mathscr{I}}$ for those coming from $\mathscr{I}_-$ in the sense of scattering theory.

This is consistent with a formal definition in terms of mode expansions, proposed by Ottewill-Winstanley in the case of scalar fields [52] and Casals-Dolan-Nolan-Ottewill-Winstanley for fermions [5], cf. [6] for electromagnetic fields. The approach via mode expansions is advantageous for practical computations. However, making the definition of the state rigorous and proving the Hadamard condition requires a sufficiently precise scattering theory (which distinguishes between solutions coming from $\mathscr{H}_-$ and solutions coming from $\mathscr{I}_-$), combined with high frequency estimates for solutions in terms of their asymptotic data. On top of that, the fact that none of the Killing vector fields is everywhere time-like makes it impossible to use the arguments employed in [12] to show the Hadamard condition.

1.2. Main result

In the present paper, we provide the first rigorous result of existence of a distinguished Hadamard state on a rotating black hole spacetime. More precisely, we give a precise definition and prove the Hadamard property of the *Unruh state* (or more pedantically, of the *past Unruh state*) in the case of massless Dirac fields on Kerr spacetime.

We consider the four-dimensional spacetime $(\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}, g)$ which is the union of the Kerr black hole exterior and interior spacetimes with rotation parameter $a > 0$ and view it as a subregion of the Kerr-Kruskal extension (M, g) (see Figure 1).

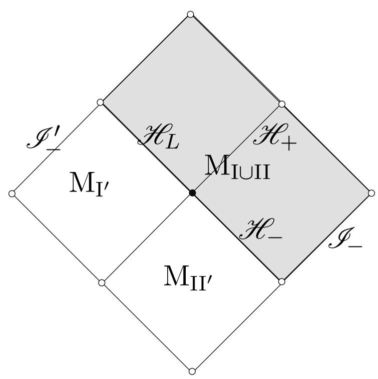

FIGURE 1. The Carter-Penrose diagram of the spacetime $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ (represented by the shaded region) embedded in the larger spacetime M. Scattering in the exterior region M_{I} refers to data at the past horizon $\mathscr{H}_- \subset \mathscr{H}_{\mathrm{L}}$ and at past null infinity $\mathscr{I}_-$. Scattering in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ refers to data at the long horizon $\mathscr{H}_{\mathrm{L}}$ and at $\mathscr{I}_-$. Scattering in M requires an extra piece of data at $\mathscr{I}'_-$.

The Dirac operator $\not{D}$ acting on smooth sections of the canonical spinor bundle $\mathcal{S}$ over M is the differential operator defined as

$$\not{D} = g^{\mu\nu}\gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu},$$

see Sections 2 and 5 for details. In the massless case considered here, it is well known that the whole analysis can be reduced to the *Weyl equation* $\mathbb{D}\phi = 0$, which accounts for half of the degrees of freedom, see Section 2.