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HODGE-RIEMANN BILINEAR RELATIONS FOR SCHUR CLASSES OF AMPLE VECTOR BUNDLES

BY JULIUS ROSS AND MATEI TOMA

ABSTRACT. – Let X be a d dimensional projective manifold, E be an ample vector bundle on X and $0 \leq \lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_1 \leq \operatorname{rk}(E)$ be a partition of $d-2$. We prove that the Schur class $s_\lambda(E) \in H^{d-2,d-2}(X)$ has the Hard Lefschetz property and satisfies the Hodge-Riemann bilinear relations. As a consequence we obtain various new inequalities between characteristic classes of ample vector bundles, including a higher-rank version of the Khovanskii-Teissier inequalities.

RÉSUMÉ. – Soient X une variété projective lisse de dimension d, E un fibré vectoriel ample sur X et $0 \leq \lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_1 \leq \operatorname{rk}(E)$ une partition de $d-2$. Nous démontrons que la classe de Schur $s_\lambda(E) \in H^{d-2,d-2}(X)$ a la propriété de Lefschetz difficile et satisfait aux relations bilinéaires de Hodge-Riemann. Nous en déduisons plusieurs nouvelles inégalités pour les classes caractéristiques des fibrés vectoriels amples y compris une version en rang supérieur des inégalités de Khovanskii-Teissier.

1. Introduction

As is well known, Hodge Theory on projective manifolds has a number of deep topological consequences. The two basic examples of this are the Hard Lefschetz Theorem which implies that if L is an ample line bundle on a projective manifold X of dimension d, and $k \leq d$ is chosen so $d-k$ is even then the map

$$H^{\frac{d-k}{2},\frac{d-k}{2}}(X;\mathbb{R}) \xrightarrow{\wedge c_1(L)^k} H^{\frac{d+k}{2},\frac{d+k}{2}}(X;\mathbb{R})$$

is an isomorphism, and the Hodge-Riemann bilinear relations which state that the bilinear form

$$(\alpha,\alpha') \mapsto (-1)^{\frac{d-k}{2}} \int_X \alpha c_1(L)^k \alpha' \quad \text{for} \quad \alpha,\alpha' \in H^{\frac{d-k}{2},\frac{d-k}{2}}(X;\mathbb{R})$$

is positive definite on the primitive cohomology

$$H_p^{\frac{d-k}{2},\frac{d-k}{2}}(X;\mathbb{R}) := \{\alpha : \alpha \wedge c_1(L)^{k+1} = 0\}.$$

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Given the importance of these results it is natural to question if these properties continue to hold when $c_1(L)^k$ is replaced by some other class in $H^{k,k}(X;\mathbb{R})$. One result in this direction is that of Bloch-Gieseker [3] which implies that if E is an ample vector bundle of rank $e \le d$ on X with $d-e$ even then $c_e(E)$ has the Hard Lefschetz property, i.e., the map

$$H^{\frac{d-e}{2},\frac{d-e}{2}}(X;\mathbb{R}) \xrightarrow{\wedge c_e(E)} H^{\frac{d+e}{2},\frac{d+e}{2}}(X;\mathbb{R})$$

is an isomorphism.

The main result of this paper extends this statement, when $e = d-2$, to show that in fact the Hodge-Riemann bilinear relations also hold for $c_e(E)$, and furthermore generalizes it to all Schur classes. This is the following

THEOREM 1.1 (= Theorem 5.3). – *Let E be a rank e ample vector bundle on a projective manifold X of dimension d, let $h \in H^{1,1}(X,\mathbb{Z})$ be an ample class and set $c_i := c_i(E)$. Given*

$$0 \le \lambda_N \le \lambda_{N-1} \le \cdots \le \lambda_1 \le e$$

with $\sum_i \lambda_i = d-2$ consider the Schur class

$$s_\lambda(E) = \det\begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+N-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+N-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{\lambda_N-N+1} & c_{\lambda_N-N+2} & \cdots & c_{\lambda_N} \end{pmatrix} \in H^{d-2,d-2}(X,\mathbb{R}).$$

Then

1. *The Hard Lefschetz Property holds for $s_\lambda(E)$. That is, the map*

$$H^{1,1}(X;\mathbb{R}) \to H^{d-1,d-1}(X;\mathbb{R}) \quad \alpha \mapsto \alpha \wedge s_\lambda(E)$$

is an isomorphism.

2. *The Hodge-Riemann bilinear relations hold for $s_\lambda(E)$. That is, the intersection pairing*

$$(\alpha,\alpha') \mapsto \int_X \alpha s_\lambda(E)\alpha' \quad \text{for} \quad \alpha,\alpha' \in H^{1,1}(X;\mathbb{R})$$

is negative definite on the primitive cohomology

$$H^{1,1}_{p,\lambda}(X;\mathbb{R}) := \{\alpha : \int_X \alpha \wedge s_\lambda(E) \wedge h = 0\}.$$

The above theorem is in the same spirit as that of Fulton-Lazarsfeld [20] who consider such Schur classes when $\sum_i \lambda_i = d$ and prove that if E is ample then $\int_X s_\lambda(E) > 0$. From this point of view one can also view Theorem 1.1 as a statement about positivity properties enjoyed by ample vector bundles.

As an application we partially answer a question posed by Debarre-Ein-Lazarsfeld-Voisin [9] (also Lehmann-Fulger [16]) concerning the relation between the cone spanned by Schur classes of nef bundles and the cone of positive higher codimensional cycles. In summary, we show that the former cone is strictly contained in the nef cone of codimension 2 cycles on the product of a very general principally polarized abelian surface with itself.

*

The classical Hodge-Riemann bilinear relations are known to imply the Hodge-Index inequality as well as many generalizations, and wrapped up in our account of Theorem 1.1 are a number of similar such inequalities. We list two now, the second of which is particularly striking.

THEOREM 1.2 (= Theorem 3.2). – *Let X be a projective manifold of dimension $d \geq 2$, let E be an ample bundle on X with* $\mathrm{rk}(E) \geq d-1$ *and let h be an ample class on X. Then for any $\alpha \in H^{1,1}(X;\mathbb{R})$*

$$\int_X \alpha^2 c_{d-2}(E) \int_X h c_{d-1}(E) \leq 2 \int_X \alpha c_{d-2}(E) h \int_X \alpha c_{d-1}(E) \tag{1.1}$$

with equality if and only if $\alpha = 0$.

Notice also that (1.1) implies that the bilinear form $(\alpha, \alpha') \mapsto \int_X \alpha c_{d-2}(E)\alpha'$ is negative definite on the subspace $\{\alpha : \int_X \alpha c_{d-1}(E) = 0\}$ (from which the Hodge-Riemann bilinear relations follow easily).

THEOREM 1.3 (= Theorem 7.4). – *Let X be a projective manifold of dimension d, let E be an ample bundle on X with* $\mathrm{rk}(E) \geq d$ *and let h be an ample class on X. Then the map*

$$i \mapsto \int_X c_i(E) h^{d-i} \quad \textit{for} \quad i = 0, \ldots, d$$

is strictly log-concave. That is, given integers $0 \leq i < j < k \leq d$ and defining t so

$$ti + (1-t)k = j$$

we have

$$t \log \int_X c_i(E) h^{d-i} + (1-t) \log \int_X c_k(E) h^{d-k} < \log \int_X c_j(E) h^{d-i}.$$

One should think of this statement as a higher rank version of the famous Khovanskii-Teissier inequalities (see Remark 7.6).

It is possible to generalize this log-concavity to other Schur classes as follows. For any partition μ the Schur polynomial $s_\mu(x_1, \ldots, x_e)$ is a symmetric polynomial, from which we may define new symmetric polynomials $s_\mu^{(i)}$ by requiring

$$s_\mu(x_1 + t, \ldots, x_e + t) = \sum_{i=0}^{|\lambda|} s_\mu^{(i)}(x_1, \ldots, x_e) t^i \quad \text{for all} \quad t \in \mathbb{R}.$$

So if $x_1, \ldots, x_e$ are the Chern roots of a bundle E on X we have characteristic classes

$$s_\mu^{(i)}(E) \in H^{|\mu|-i,|\mu|-i}(X;\mathbb{R}).$$

THEOREM 1.4 (= Theorem 7.4). – *Let X be a projective manifold of dimension d, let E be an ample bundle on X with $e =$* $\mathrm{rk}(E)$ *$\geq d$ and let h be an ample class on X. Also let $0 \leq \mu_N \leq \cdots \leq \mu_1 \leq e$ be a partition of e.*

Then the map

$$i \mapsto \int_X s_\mu^{(e-i)}(E) h^{d-i} \quad \textit{for} \quad i = 0, \ldots, d$$

is strictly log-concave.