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EQUIVARIANT LANDAU-GINZBURG MIRROR SYMMETRY

BY Jérémy GUÉRÉ

ABSTRACT. – We use the classical localization formula of Atiyah-Bott to express Hodge integrals for the quantum singularity (FJRW) theory of chain polynomials and we obtain the first equivariant version of mirror symmetry without concavity, generalizing the work of Chiodo-Iritani-Ruan on the Landau-Ginzburg side. Furthermore, we phrase our proof in a general framework that is suitable for future studies of gauged linear sigma models (GLSM).

RÉSUMÉ. – Nous utilisons la formule de localisation d'Atiyah-Bott afin d'exprimer les intégrales de Hodge de la théorie quantique des singularités (théorie FJRW) de type chaine. Nous obtenons ainsi la première version équivariante de la symétrie miroir en l'absence de l'hypothèse de concavité, généralisant le côté Landau-Ginzburg du travail de Chiodo-Iritani-Ruan. De plus, nous formalisons notre preuve dans un langage propre à l'étude future des sigma-modèles de jauge linéaire (GLSM).

0. Introduction

Gromov-Witten theory has known a tremendous development in the last thirty years. Taking its origins in theoretical physics, it is mathematically formulated as an intersection theory of complex curves traced on a complex smooth projective variety, and it gives invariants that one thinks of as a virtual count of these curves. The most famous example is a full computation of the genus-zero invariants enumerating rational curves on the quintic threefold Candelas,Gi,LLY.

Recently, a partial generalization of Gromov-Witten theory, known as Gauge Linear Sigma Model [10], has given the opportunity to develop enumerative geometry for other mathematical objects. The quantum singularity theory, also known as Fan-Jarvis-Ruan-Witten (FJRW) theory [9, 11, 17], is one such instance.

In our previous work [13], we gave the first genus-zero computation of the virtual cycle in the quantum singularity (FJRW) theory in a range of cases where the state-of-the-art techniques relying on the concavity condition did not apply. As an application, we proved a mirror symmetry theorem for these theories.

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Later in [14], we generalized our results and obtained the first all-genus computation on the moduli space of Landau-Ginzburg spin curves, providing we first cap the virtual cycle with the Euler class of the Hodge vector bundle. It led to Hodge integral calculations in the quantum singularity (FJRW) theory in a range of cases where the techniques relying on Teleman's reconstruction theorem for generically semi-simple Cohomological Field Theories (CohFTs) did not apply. As an application, we proved in [3] the DR/DZ conjecture for 3-spin, 4-spin, and 5-spin theories, that is, the equivalence of the Double Ramification (DR) hierarchy with the Dubrovin-Zhang (DZ) hierarchy for these theories.

Interestingly, there are up to now no counterparts of [13, 14] for Gromov-Witten theory of hypersurfaces in weighted projective spaces, although such a parallel story should appear under the light of Landau-Ginzburg/Calabi-Yau correspondence [5]. Genus-zero Gromov-Witten invariants of hypersurfaces in weighted projective spaces are still unknown, as soon as the convexity condition fails. Even for projective hypersurfaces, there is no general description of Hodge integrals in positive genus.

Both papers [13, 14] relied on a new technique based on the notion of recursive complexes and it has been our main focus for the last five years to understand how to carry this technique into the Gromov-Witten side. To achieve this ambitious goal, we shed new light on our previous results by changing our strategy to a more Gromov-Witten-like approach: we make use of the localization formula of Atiyah-Bott [1], developped in the algebraic category by [7, 8], to carry the computation of Hodge integrals in FJRW theory. We then give a new and shorter proof of the results in [13, 14]. We also upgrade our mirror theorem [13, Theorem 4.4] to an equivariant version of it, in the spirit of [5, Section 4.3], see Theorem 2.10. We highlight the fact it was out of reach with the previous technique.

Importantly, we phrase our new method in a very general framework that is relevant when working with Landau-Ginzburg models. We thus believe it is suitable for the study of any Gauged Linear Sigma Model (GLSM) [10]. Indeed, following our work [6, Section 6], we see that the definition of virtual cycles in hybrid GLSMs can be phrased as a localized Chern character of a two-periodic complex on a big moduli space denoted $\square$ in [6], and that the picture in [6, Section 1.5] is a special case of the one we describe in Section 1. Furthermore, it is worth noticing that Gromov-Witten theory of a complete intersection in a toric Deligne-Mumford (DM) stack is a special instance of a GLSM via the comparison [16]. In particular, it is absolutely clear that the strategy developed in Section 1 applies, with little changes compared to Section 2, to the case of hypersurfaces in weighted projective spaces which are defined by chain polynomials. However, writing this paper, we discovered a more direct way to pursue this goal and we decided to leave this result to another paper [15].

Acknowledgement. – The author is grateful to Alexander Polishchuk who suggested first to look for a proof of the results in [13, 14] using the localization formula.

1. Localization formula for localized Chern characters

Here, we describe the localization method that we apply to FJRW theory in the next section. We explain it in a more general framework, so that it can serve as a reference for future works regarding GLSM models.

1.1. Localized Chern character

We work over an arbitrary field $\mathbb{K}$ and we consider the following set-up:

$$\begin{tikzcd} T \arrow[dr, dashed] & & & V \arrow[dl, dashed] \\ & Y \arrow[r, hook, "j"] & X \arrow[d, "p"'] & \\ & & S & E \arrow[l, dashed] \end{tikzcd} \quad \circlearrowleft \mathbb{K}^*$$

where S is a proper DM stack, X is a DM stack over S, the substack Y is a local complete intersection in X and is proper over S, and V, E, and T are locally free sheaves (vector bundles) over the DM stacks X, S, and Y. Moreover, we have an action of the multiplicative group $\mathbb{K}^*$ on the fibers of p; precisely an action on X and on S such that the action on S is trivial and the projection morphism $p\colon X \to S$ is equivariant. We assume the closed substack Y to be $\mathbb{K}^*$-invariant and the vector bundles V, E, and T to be $\mathbb{K}^*$-equivariant. Furthermore, we assume the $\mathbb{K}^*$-fixed locus of X to be a closed substack of Y that we denote Y_F.

We also consider four global sections

$$\alpha, \alpha' \in H^0(X, V^\vee)\ ,\ \alpha'' \in H^0(X, p^*E)\ ,\ \text{and } \beta \in H^0(X, V)$$

such that α, α'', and β are $\mathbb{K}^*$-equivariant but α' is not, and that

$$\beta(\alpha) = \beta(\alpha') = 0.$$

For every $t_1, t_2 \in \mathbb{K}$, we define global sections

$$\alpha(t_1) = \alpha + t_1\alpha' \in H^0(X, V^\vee) \quad \text{and} \quad \widetilde{\alpha}(t_1, t_2) = \alpha + t_1\alpha' + t_2\alpha'' \in H^0(X, V^\vee \oplus p^*E)$$

and Koszul two-periodic complexes

$$K(t_1) = \big(\Lambda^\bullet(V^\vee), \delta(t_1)\big) \quad \text{and} \quad \widetilde{K}(t_1, t_2) = \Big(\Lambda^\bullet(V^\vee \oplus p^*E), \widetilde{\delta}(t_1, t_2)\Big)$$

over the DM stack X, where the maps are

$$\delta(t_1) = \alpha(t_1) \wedge \cdot + \beta(\cdot) \quad \text{and} \quad \widetilde{\delta}(t_1, t_2) = \widetilde{\alpha}(t_1, t_2) \wedge \cdot + \beta(\cdot).$$

We observe the following

$$\begin{aligned} K(t_1) \text{ is } \mathbb{K}^*\text{-equivariant} &\iff t_1 = 0, \\ \widetilde{K}(t_1, t_2) \text{ is } \mathbb{K}^*\text{-equivariant} &\iff t_1 = 0, \end{aligned}$$

and we have the equality of two-periodic complexes

$$\widetilde{K}(t_1, 0) = K(t_1) \otimes \Lambda^\bullet\left(p^*E\right). \tag{1}$$

Furthermore, we assume

$$\begin{aligned} K(t_1) \text{ is strictly exact off } Y &\iff t_1 \neq 0, \\ \widetilde{K}(t_1, t_2) \text{ is strictly exact off } Y &\iff (t_1, t_2) \neq 0. \end{aligned}$$

We recall from [18] that a two-periodic complex is strictly exact off Y if it is exact off Y and the images of the maps are subbundles.