ISSN 0012-9593

quatrième série - tome 56 fascicule 1 janvier-février 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Michael HARRIS & Chandrashekhar B. KHARE & Jack A. THORNE

A local Langlands parameterization for generic supercuspidal representations of p-adic G_2

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 56, 2023, p. 257 à 286

A LOCAL LANGLANDS PARAMETERIZATION FOR GENERIC SUPERCUSPIDAL REPRESENTATIONS OF p-ADIC G_2

BY Michael HARRIS, Chandrashekhar B. KHARE
AND Jack A. THORNE
With an appendix by Gordan SAVIN

ABSTRACT. – We construct a Langlands parameterization of supercuspidal representations of G_2 over a p-adic field. More precisely, for any finite extension $K/\mathbb{Q}_p$ we will construct a bijection

$$\mathcal{L}_g : \mathcal{A}^0_g(G_2, K) \longrightarrow \mathcal{G}^0(G_2, K)$$

from the set of *generic* supercuspidal representations of $G_2(K)$ to the set of irreducible continuous homomorphisms $\rho : W_K \to G_2(\mathbb{C})$ with W_K the Weil group of K. The construction of the map is simply a matter of assembling arguments that are already in the literature, together with a previously unpublished theorem of G. Savin on exceptional theta correspondences, included as an appendix. The proof that the map is a bijection is arithmetic in nature, and specifically uses automorphy lifting theorems. These can be applied thanks to a recent result of Hundley and Liu on automorphic descent from GL(7) to G_2.

RÉSUMÉ. – Nous construisons une paramétrisation de Langlands des représentations supercuspidales de G_2 sur un corps p-adique. Plus précisément, pour chaque extension $K/\mathbb{Q}_p$ nous construisons une application bijective

$$\mathcal{L}_g : \mathcal{A}^0_g(G_2, K) \longrightarrow \mathcal{G}^0(G_2, K)$$

de l'ensemble des représentations supercuspidales *génériques* de $G_2(K)$ vers l'ensemble des morphismes continus et irréductibles $\rho : W_K \to G_2(\mathbb{C})$, où W_K désigne le groupe de Weil de K. Pour construire cette application il suffit de combiner des arguments qui sont déjà dans la littérature, plus un théorème inédit de G. Savin sur les correspondances thêta exceptionnelles, qui est démontré dans un appendice écrit par ce dernier. La démonstration de la bijectivité de l'application est de nature arithmétique, et utilise notamment des théorèmes de relèvement automorphes. Ceux-ci s'appliquent à notre problème grâce à un résultat récent de Hundley et Liu sur la descente automorphe de GL(7) vers G_2.

M.H. was partially supported by NSF Grant DMS-1701651. C.K. was partially supported by NSF Grant DMS-2200390. J.T.'s work received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 714405).

doi:10.24033/asens.2533

Introduction

The purpose of this article is to construct a Langlands parameterization of supercuspidal representations of G_2 over a p-adic field. More precisely, for any finite extension $K/\mathbb{Q}_p$ we will construct a bijection

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \to \mathcal{G}^0(G_2, K)$$

from the set of generic supercuspidal representations of $G_2(K)$ to the set of irreducible continuous homomorphisms $\rho : W_K \to G_2(\mathbb{C})$ with W_K the Weil group of K (more precisely, the bijection is between sets of equivalence classes). The construction of the map is simply a matter of assembling arguments that are already in the literature; the article [26] effectively contains the construction, although it doesn't specifically point out the application to supercuspidal representations. The proof of surjectivity is an application of a recent result of Hundley and Liu [24], which allows us to carry out a strategy, based on automorphy lifting theorems, that was initially developed in [4] as an application of Vincent Lafforgue's global parameterization of automorphic representations over function fields. The proof of injectivity also uses global arithmetic methods, including automorphy lifting theorems and the Ramanujan conjecture for self-dual, regular algebraic automorphic representations of $\mathrm{GL}(n)$, alongside known results on liftings (especially [36, 46]).

The parameterization is constructed in two steps. First, following [17, 11, 36], among other references, we use the exceptional dual reductive pair $(G_2, \mathrm{PGSp}(6))$ in E_7 to define local and global correspondences from representations of G_2 to representations of $\mathrm{PGSp}(6)$. We then lift to $\mathrm{Sp}(6)$ and use the functorial transfer of [8] to obtain an automorphic representation of $\mathrm{GL}(7)$. Using the local Langlands correspondence for $\mathrm{GL}(n)$, we can thus obtain a parameterization of supercuspidal representations of G_2 by Galois parameters with values in $\mathrm{GL}(7)$. We use a global argument and Chebotarev density (following [6]) to show that the parameter takes values in the image of G_2 under its 7-dimensional irreducible representation r_7.

The proof of surjectivity is arithmetic. For the moment, let K be a p-adic field and let ρ be a continuous homomorphism

$$\rho : W_K \to G_2(\mathbb{C}).$$

We assume ρ is *irreducible*: that its image is contained in no proper parabolic subgroup. Since the image is finite, we may replace the coefficient field $\mathbb{C}$ by a sufficiently large finite field k of characteristic $\ell \neq p$. Following Moret-Bailly we show first that K may be viewed as the completion at a p-adic place v of a totally real field F, and that ρ can be extended to a surjective homomorphism $\mathrm{Gal}(\overline{F}/F) \to G_2(k)$ that is *odd*, in an appropriate sense. We then use the lifting method in [27] to lift ρ to a homomorphism $\tilde{\rho} : \mathrm{Gal}(\overline{F}/F) \to G_2(W(k))$ in such a way that $r_7 \circ \tilde{\rho}$ is geometric, in the sense of Fontaine-Mazur, and Hodge-Tate regular.

Now we can apply automorphy lifting theorems, as in [3], to show that $r_7 \circ \tilde{\rho}$ is potentially automorphic – that its restrictions to appropriate totally real Galois extensions F'/F are attached to a cuspidal cohomological self-dual automorphic representation π' of $\mathrm{GL}(7, \mathbf{A}_{F'})$. Choosing F' carefully, we can then descend π' to an automorphic representation π'' of $\mathrm{GL}(7, \mathbf{A}_{F''})$ over the fixed field F'' of a decomposition group $\mathrm{Gal}(F'_{v'}/F_v) \subset \mathrm{Gal}(F'/F)$. At this point we apply the result of Hundley and Liu to show that π'' is in the image of the functorial transfer from $G_2(\mathbf{A}_{F''})$ to $\mathrm{GL}(7, \mathbf{A}_{F''})$ of an automorphic representation Π of $G_2(\mathbf{A}_{F''})$, and we conclude by observing that the local component Π_v is supercuspidal and has parameter ρ. As a bonus, the construction of [24] provides a globally generic Π, so we see that ρ is the parameter of a generic supercuspidal representation.

There has been a good deal of work on the local representation theory as well as on the automorphic theory of G_2. Notably, the articles [11], [37], and [36] come very close to establishing a complete local Langlands correspondence for G_2 and to relate the correspondence to the exceptional theta correspondence used here (1) ; the article [24] comes very close to characterizing the image of functoriality from G_2 to $\mathrm{GL}(7)$. The purpose of this article is not to replace the articles just cited – indeed, the results of these articles are used crucially in the proof of our main theorem – but rather to illustrate the possibility of applying a combination of arithmetic and automorphic methods to the local correspondence.

Acknowledgements

As mentioned above, this paper implements a strategy that was developed in our joint paper with Gebhard Böckle, and we are grateful to him for many discussions. Thanks are due to Joseph Hundley and Baiying Liu for bringing to our attention their recent result on descent for G_2, on which our argument crucially depends. We also thank Jeff Adams, Wee Teck Gan, Dihua Jiang, Aaron Pollack, and Gordan Savin for help with references, and Freydoon Shahidi for pointing out an error in the L-function calculation in the original proof of Proposition 4.5. We thank Savin for agreeing to write the appendix that proves a global genericity result that allows us to define the local parameterization unambiguously.

We also thank the anonymous referees for their very careful reading, and for their many suggestions that, we hope, have made the text more readable.

Notation

If K is a perfect field, we will write Γ_K for its Galois group relative to a fixed algebraic closure. When K is a number field, we will fix an algebraic closure $\overline{K}/K$, algebraic closures $\overline{K}_v/K_v$ for each place v of K, and embeddings $\overline{K} \to \overline{K}_v$ extending $K \to K_v$. These choices determine embeddings $\Gamma_{K_v} \to \Gamma_K$ for each place v of K. If v is a finite place, then $I_{K_v} \subset \Gamma_{K_v}$ is the inertia group.

We write $\epsilon : \Gamma_K \to \mathbb{Z}_\ell^\times$ for the ℓ-adic cyclotomic character. By abuse of notation, we also write ϵ for the pushforward of this character to the group of units of any $\mathbb{Z}_\ell$-algebra.

If F is a totally real number field, n is an *odd* integer, and π is a cuspidal, regular algebraic automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$ which is self-dual, in the sense that $\pi \cong \pi^\vee$, then for any isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ there is an associated semi-simple ℓ-adic Galois representation $r_\iota(\pi) : \Gamma_F \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$. This is characterized, up to isomorphism, by

(1) While revising this article we learned of the new preprint [12] of Gan and Savin that establishes Howe duality as well as a dichotomy result for these exceptional theta correspondences. It is likely that some of the arguments in the present paper can now be simplified and many of the references can be consolidated. In the meantime, Gan and Savin have released a new paper [13] containing a complete proof of the local Langlands correspondence for G_2 over a p-adic field. The correspondence of [13], like the parametrization for generic supercuspidals given here, is compatible with global correspondences. The methods of [13] are quite different from the ones presented here.