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# SYMPLECTIC LEAVES FOR GENERALIZED AFFINE GRASSMANNIAN SLICES

BY DINAKAR MUTHIAH AND ALEX WEEKES

ABSTRACT. – The generalized affine Grassmannian slices $\overline{\mathcal{W}}^{\lambda}_{\mu}$ are algebraic varieties introduced by Braverman, Finkelberg, and Nakajima in their study of Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories. We prove a conjecture of theirs by showing that the dense open subset $\mathcal{W}^{\lambda}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$ is smooth. An explicit decomposition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$ into symplectic leaves follows as a corollary. Our argument works over an arbitrary ring and in particular implies that the complex points $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ form a smooth holomorphic symplectic manifold. A subtle aspect of the method is our essential use of ind-schemes that are formally smooth but not smooth.

RÉSUMÉ. – Les tranches dans les grassmanniennes affines généralisées $\overline{\mathcal{W}}^{\lambda}_{\mu}$ sont des variétés affines introduites par Braverman, Finkelberg et Nakajima au cours de leur étude des branches de Coulomb pour les théories de jauge $3d$ $\mathcal{N} = 4$ de type carquois. Nous prouvons une de leurs conjectures, en montrant que l'ouvert affine $\mathcal{W}^{\lambda}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$ est lisse. Une décomposition précise en feuilles symplectiques en découle. Notre preuve est valable sur un anneau arbitraire et, en particulier, nous montrons que l'ensemble des points complexes $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ forme une variété complexe. Un aspect subtil de notre méthode est l'emploi essentiel des ind-schémas qui sont formellement lisses mais qui ne sont pas lisses.

## 1. Introduction

Affine Grassmannian slices for a reductive group $G$ are objects of considerable interest. As transversal slices to spherical Schubert varieties, they capture information about singularities in the affine Grassmannian. By the geometric Satake correspondence, these singularities are known to carry representation-theoretic information about the Langlands dual group of $G$. Additionally, they have a Poisson structure that quantizes to the truncated shifted Yangians [11]. Furthermore, they form a large class of conical symplectic singularities that do not admit symplectic resolutions in general. As such, they form an important test ground for ideas in symplectic algebraic geometry and representation theory.

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Recently, Braverman, Finkelberg, and Nakajima [2] showed that affine Grassmannian slices arise as Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories. Their construction of affine Grassmannian slices is particularly satisfying because: (1) the quantization comes essentially for free in their construction, (2) their construction works for arbitrary symmetric Kac-Moody type. Because of point (2), the Coulomb branch perspective seems to be a fruitful path toward understanding the geometric Satake correspondence in affine type and beyond (see e.g., [6, 15]).

However, their construction produces more than just affine Grassmannian slices: usual affine Grassmannian slices are indexed by a pair of dominant coweights $\lambda$ and $\mu$, but their construction does not constrain $\mu$ to be dominant. Rather, they construct *generalized affine Grassmannian slices* denoted $\overline{\mathcal{W}}^{\lambda}_{\mu}$ for $\lambda$ constrained to be dominant but for *arbitrary* $\mu \leq \lambda$.

The geometry of the generalized affine Grassmannian slices for $\mu$ non-dominant is less understood than the case of $\mu$ dominant. For example, there is a disjoint decomposition

$$\overline{\mathcal{W}}^{\lambda}_{\mu} = \bigsqcup_{\substack{\nu \text{ dominant}, \\ \mu \leq \nu \leq \lambda}} \mathcal{W}^{\nu}_{\mu} \tag{1.1}$$

that Braverman, Finkelberg and Nakajima conjecture ([2, Remark 3.19]) to be a decomposition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$ into symplectic leaves. They show that this would follow if one could show that the subvarieties $\mathcal{W}^{\lambda}_{\mu}$ are smooth for all $\lambda$ and $\mu$. In this note, we show the following, which proves their conjecture.

THEOREM 1.2 (Corollary 3.17). – *For any $\lambda \geq \mu$ with $\lambda$ dominant, the variety $\mathcal{W}^{\lambda}_{\mu}$ is smooth.*

In particular, it follows that the set of $\mathbb{C}$-points $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ is a smooth holomorphic symplectic manifold. This verifies part of an expectation that it is also hyper-Kähler, since $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ should be identified with a moduli space of singular monopoles on $\mathbb{R}^3$, see [2], [4].

### 1.1. Previously known cases

Theorem 1.2 is known when $\mu$ is dominant because in this case $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is a usual affine Grassmannian slice. It is also known for $\mu \leq w_0(\lambda)$ where $w_0$ is the longest element of the Weyl group [2, Remark 3.19]. In type A, all cases are known by work of Nakajima and Takayama on Cherkis bow varieties [16, Theorem 7.13]. In [13], Krylov and Perunov prove Theorem 1.2 in general type for $\lambda$ minuscule and $\mu$ lying in the orbit of $\lambda$ under the Weyl group. In fact, they prove more: they show that $\overline{\mathcal{W}}^{\lambda}_{\mu} = \mathcal{W}^{\lambda}_{\mu}$ is an affine space in this case.

We note that our main theorem has been expected by physicists, since $\mathcal{W}^{\lambda}_{\mu}$ should be a space of singular instantons as mentioned above, and that the decomposition (1.1) is an instance of *monopole bubbling*. We refer the reader again to [4], and to the references in the physics literature given in the introduction of [2], as well as in [14].

Finally, generalized affine Grassmannian slices of the form $\mathcal{W}^{0}_{\mu}$ had been previously considered in a different guise: these are the so called "open Zastava spaces" whose smoothness is deduced by a certain cohomology vanishing computation [7]. Our approach gives a direct group-theoretic proof of this smoothness. It would be interesting to understand how these two approaches are precisely related. We elaborate on this briefly in §3.3.3.

### 1.2. Our approach

There is a group theoretic construction of generalized affine Grassmannian slices $\overline{\mathcal{W}}^{\lambda}_{\mu}$ and their open subschemes $\mathcal{W}^{\lambda}_{\mu}$ given in [2, Section 2(xi)], inspired by the scattering matrix description of singular monopoles from [4, Section 6.4.1]. We slightly modify this group-theoretic construction to produce spaces that we call $\overline{\mathcal{X}}^{\lambda}_{\mu}$ and $\mathcal{X}^{\lambda}_{\mu}$. We show these spaces are products of the corresponding $\mathcal{W}$-versions and an infinite dimensional affine space (Proposition 3.8). We then show that the spaces $\mathcal{X}^{\lambda}_{\mu}$ are formally smooth (Theorem 3.14), from which we conclude that the spaces $\mathcal{W}^{\lambda}_{\mu}$ are formally smooth. Because the $\overline{\mathcal{W}}^{\lambda}_{\mu}$ (and hence $\mathcal{W}^{\lambda}_{\mu}$) are known to be finitely presented (see Proposition 2.10 for a direct group-theoretic proof), we conclude that $\mathcal{W}^{\lambda}_{\mu}$ is in fact smooth. A subtle point in our approach is that we make use of the formal smoothness of an ind-scheme $\mathcal{X}^{\lambda}$ that is *not* smooth, so the use of infinite-dimensional spaces and formal smoothness appears essential in our approach (see Remark 3.10).

We note that our approach to smoothness is analogous to a standard approach to the smoothness of usual affine Grassmannian slices (and in fact general Schubert slices for partial flag varieties, see e.g.,[12, §1.4])). Our space $\mathcal{X}^{\lambda}_{\mu}$ is constructed using an auxiliary space $\mathcal{X}_{\mu}$ that plays the role of an open chart in the affine Grassmannian. We explain this briefly in §3.3.2.

### 1.3. Acknowledgements

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## 2. Preliminaries

### 2.1. Schemes and functors

Let $\Bbbk$ be a commutative ring. Let $\mathrm{Alg}_{\Bbbk}$ be the category of commutative $\Bbbk$-algebras, and let $\mathrm{Sch}_{\Bbbk}$ be the category of $\Bbbk$-schemes. We define the category $\mathrm{Func}_{\Bbbk}$ of $\Bbbk$-functors to be the category of functors $\mathrm{Alg}_{\Bbbk} \to \mathrm{Set}$. Recall that there is a fully-faithful embedding $\mathrm{Sch}_{\Bbbk} \hookrightarrow \mathrm{Func}_{\Bbbk}$ coming from the Yoneda Lemma and the fact that morphisms of schemes are local for the Zariski topology. Following the usual terminology, we will call this inclusion the map sending a scheme to the functor it represents. All the functors we consider will be ind-schemes (of possibly ind-infinite type), so it is not strictly necessary to consider them as functors. However, we will be focused on questions of formal smoothness, so the functorial viewpoint will be essential.