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# THE PLANCHEREL FORMULA FOR COMPLEX SEMISIMPLE QUANTUM GROUPS

BY CHRISTIAN VOIGT AND ROBERT YUNCKEN

ABSTRACT. – We calculate the Plancherel formula for complex semisimple quantum groups, that is, Drinfeld doubles of $q$-deformations of compact semisimple Lie groups. As a consequence we obtain a concrete description of their associated reduced group $C^*$-algebras. The main ingredients in our proof are the Bernstein-Gelfand-Gelfand complex and the Hopf trace formula.

RÉSUMÉ. – Nous calculons la formule de Plancherel pour les groupes quantiques semi-simples complexes, c'est-à-dire les doubles de Drinfeld des $q$-déformations des groupes de Lie semi-simples compacts. En conséquence nous obtenons une description concrète des $C^*$-algèbres réduites de groupe associées. Les ingrédients principaux dans notre preuve sont le complexe de Bernstein-Gelfand-Gelfand et la formule des traces de Hopf.

## 1. Introduction

Complex semisimple quantum groups are locally compact quantum groups which were constructed and first studied by Podleś and Woronowicz [19]. They are defined as Drinfeld doubles of $q$-deformations of compact semisimple Lie groups, and can be viewed as deformations of the corresponding complex Lie groups in a natural way. Motivated by physical considerations, Podleś and Woronowicz focused mainly on the case of the quantum Lorentz group, that is, the Drinfeld double of $\mathrm{SU}_q(2)$. It became clear later that the theory of more general complex semisimple quantum groups is linked with a range of seemingly unrelated problems in noncommutative geometry, operator $K$-theory, and the theory of $C^*$-tensor categories and subfactors, see for instance [1], [18], [23].

In the present paper we study the reduced unitary dual of complex semisimple quantum groups, and our main result is an explicit computation of the Plancherel formula. This generalizes the work of Buffenoir and Roche [5] on the quantum Lorentz group. The formula we

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obtain can be interpreted as a deformation of the Plancherel formula for the corresponding classical groups, but our method of proof is completely different.

We note that the abstract Plancherel theorem for locally compact quantum groups was established by Desmedt [7], in analogy to the classical theory. In the case of complex semisimple quantum groups the Plancherel theorem involves so-called Duflo-Moore operators because the dual Haar weights fail to be traces. This is analogous to the situation for non-unimodular locally compact groups treated by Duflo and Moore in [9]. A classical locally compact group is unimodular if and only if the Haar weight on its group $C^*$-algebra is tracial. In the quantum setting, traciality of the dual Haar weights implies unimodularity, but the converse does not hold in general. This is a well-known phenomenon which already shows up in the theory of compact quantum groups.

Given a locally compact group or quantum group, a key problem is to calculate the Plancherel formula, that is, to determine explicitly the Plancherel measure and Duflo-Moore operators in terms of a given parametrization of the unitary dual. Before describing our proof strategy in the case of complex quantum groups, let us briefly recall the approach to compute the Plancherel formula for classical complex semisimple Lie groups due to Harish-Chandra [11], see also Section 6.1 in [22]. Firstly, the characters of principal series representations are shown to be related to orbital integrals using Fourier transform. In a second step, orbital integrals on the group are transported to the Lie algebra. The final ingredient in the argument is the limit formula for orbital integrals on the Lie algebra, which in combination with the Weyl integration formula completes the proof.

Trying to adapt this strategy to the quantum case seems difficult for various reasons. In fact, it is not even clear how to define a suitable notion of orbital integrals in this setting, and there is no good analogue of the Lie algebra. We proceed by explicitly writing down candidates for the Plancherel measure and Duflo-Moore operators instead, generalizing the ones in [5]. In order to verify that our choices are correct, we determine the characters of principal series representations and define a certain linear functional on the algebra of functions on the quantum group, which we call Plancherel functional. According to the Plancherel inversion formula it then suffices to show that the Plancherel functional agrees with the counit.

For this, in turn, we use the BGG complex for quantized universal enveloping algebras, first described in [20], [10], [17], and carefully studied by Heckenberger and Kolb in [12]. In its original form, the BGG complex is a resolution of a finite dimensional representation by direct sums of Verma modules, but here we will be interested in the geometric version, which is classically presented as a differential complex of sections of induced line bundles over a flag variety. Our BGG complex is the resolution of the trivial representation by parabolically induced representations, and is obtained from the algebraic BGG complex via the category equivalence between category $\mathcal{O}$ and the category of Harish-Chandra modules due to Joseph and Letzter [14], see also [13], [24]. The key fact that allows us to compute the Plancherel functional is that its values can be identified with Lefschetz numbers of certain endomorphisms of the BGG complex. Since the BGG complex has almost trivial homology, an application of the Hopf trace formula completes the proof.

Our result shows in particular that the Plancherel measure of complex semisimple quantum groups is supported on the space of unitary principal series representations, in analogy with the classical situation. This allows us to identify the reduced group $C^*$-algebras of these quantum groups explicitly with certain continuous bundles of algebras of compact operators. As a consequence, one obtains a very transparent illustration of the deformation aspect in the operator algebraic approach to complex semisimple quantum groups, a feature which is not at all visible from the Drinfeld double construction.

Let us now explain how the paper is organized. In Section 2 we collect some preliminaries on quantum groups and fix our notation. Section 3 covers more specific background on complex semisimple quantum groups and their representations. We introduce our candidate Duflo-Moore operators for these quantum groups and compute the corresponding twisted characters of unitary principal series representations. In Section 4 we recall the abstract Plancherel Theorem for locally compact quantum groups due to Desmedt. Section 5 contains our main result, that is, the Plancherel formula for complex semisimple quantum groups. As already indicated above, the proof involves the BGG complex, and we review the necessary background material along the way. In Section 6 we make some further comments and discuss a slightly different, more direct proof of the Plancherel formula in the simplest special case of the quantum Lorentz group. This argument is considerably shorter than the original proof by Buffenoir and Roche. Finally, in Section 7 we apply the Plancherel formula to obtain an explicit description of the reduced group $C^*$-algebras of arbitrary complex semisimple quantum groups.

Let us conclude with some remarks on notation. The algebra of adjointable operators on a Hilbert space or Hilbert module $\mathcal{E}$ is denoted by $\mathbb{L}(\mathcal{E})$, and we write $\mathbb{K}(\mathcal{E})$ for the algebra of compact operators. Depending on the context, the symbol $\otimes$ denotes the algebraic tensor product over the complex numbers, the tensor product of Hilbert spaces, or the minimal tensor product of $C^*$-algebras.

## 2. Preliminaries

In this section we review some background material on quantum groups and fix our notation. For more details we refer to [6], [15], [16], [24].

Throughout we assume that our definition parameter $q$ is a strictly positive real number and $q \neq 1$. We write

$$[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$$

for the $q$-number associated with $z \in \mathbb{C}$ and use standard definitions and notation from $q$-calculus.

Let $G$ be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of a maximal compact subgroup $K \subset G$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a maximal torus $T$ of $K$ with Lie algebra $\mathfrak{t}$ such that $\mathfrak{t} \subset \mathfrak{h}$. Let us denote by $\Sigma = \{\alpha_1, \dots, \alpha_N\}$ a set of simple roots for $\mathfrak{g}$, and let $( \ , \ )$ be the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that the shortest root $\alpha$ of $\mathfrak{g}$ satisfies $(\alpha, \alpha) = 2$. The simple coroots are given by $\alpha_i^\vee = d_i^{-1}\alpha_i$ where $d_i = (\alpha_i, \alpha_i)/2$, and the entries of the Cartan matrix of $\mathfrak{g}$ are $a_{ij} = (\alpha_i^\vee, \alpha_j)$. We write $\varpi_1, \dots, \varpi_N$ for the fundamental weights, defined by stipulating $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$. Moreover we denote by $\mathbf{Q} \subset \mathbf{P} \subset \mathfrak{h}^*$ the root and