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POLAR DECOMPOSITION OF SEMIGROUPS GENERATED BY NON-SELFADJOINT QUADRATIC DIFFERENTIAL OPERATORS AND REGULARIZING EFFECTS

BY PAUL ALPHONSE AND JOACKIM BERNIER

ABSTRACT. – We characterize geometrically the regularizing effects of the semigroups generated by accretive non-selfadjoint quadratic differential operators. As a byproduct, we establish the subelliptic estimates enjoyed by these operators, being expected to be optimal. These results prove conjectures by M. Hitrik, K. Pravda-Starov and J. Viola. The proof relies on a new representation of the polar decomposition of these semigroups. In particular, we identify the selfadjoint part as the evolution operator generated by the Weyl quantization of a time-dependent real-valued nonnegative quadratic form for which we prove a sharp anisotropic lower bound.

RÉSUMÉ. – On caractérise géométriquement les effets régularisants des semi-groupes engendrés par les opérateurs quadratiques accrétifs. En corollaire, on déduit des inégalités sous-elliptiques associées à ces opérateurs, considérées comme optimales. Ces résultats démontrent des conjectures de M. Hitrik, K. Pravda-Starov et J. Viola. La démonstration repose sur une nouvelle représentation de la décomposition polaire de ces semi-groupes. En particulier, on écrit la partie autoadjointe comme un opérateur d'évolution engendré par la quantification de Weyl d'une forme quadratique positive dépendante du temps, pour laquelle on établit une minoration anisotrope précise.

1. Introduction

We consider the semigroups generated by accretive non-selfadjoint quadratic differential operators. They are the evolution operators associated with partial differential equations of the form

$$\begin{cases} \partial_t u + q^w(x, D_x)u = 0, \\ u(0, \cdot) = u_0, \end{cases} \tag{1.1}$$

where $u_0 \in L^2(\mathbb{R}^n)$, $n \geq 1$ is a fixed number and $q^w(x, D_x)$ is the *Weyl quantization* of a complex-valued quadratic form $q : \mathbb{R}^{2n} \to \mathbb{C}$ with a nonnegative real part. Denoting by

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$Q \in S_{2n}(\mathbb{C})$ the matrix of q in the canonical basis of $\mathbb{R}^{2n}$, $q^w(x, D_x)$ is nothing but the differential operator

$$q^w(x, D_x) = \begin{pmatrix} x & -i\nabla \end{pmatrix} Q \begin{pmatrix} x \\ -i\nabla \end{pmatrix}.$$

This operator is equipped with the domain

$$D(q^w) = \{u \in L^2(\mathbb{R}^n) : q^w(x, D_x)u \in L^2(\mathbb{R}^n)\}.$$

Note that this definition coincides with the classical definition of $q^w(x, D_x)$ as an oscillatory integral. We recall that since the real part of the quadratic form q is nonnegative, the quadratic operator $q^w(x, D_x)$ is shown in [23, pp. 425-426] to be maximal accretive and to generate a strongly continuous contraction semigroup $(e^{-tq^w})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$.

In this paper, proving a conjecture of M. Hitrik, K. Pravda-Starov and J. Viola in [19], we characterize and quantify geometrically the regularizing effects of $(e^{-tq^w})_{t\geq 0}$ in the asymptotic $0 < t \ll 1$. Basically, we determine how smooth and localized are the mild (i.e., semigroup) solutions of (1.1). This problematic is natural and interesting in itself but it is also motivated by its applications in control theory (see Remark 2.9 below). Furthermore, it is not trivial because, since our operators are non-selfadjoint, we have to deal with nonlinear interactions between phenomena of diffusions and transports (understood in some very weak senses). For example, considering the Kolmogorov operator $x_2\partial_{x_1} - \partial_{x_2}^2$, it can be proven that its associated semigroup is smoothing super-analytically both with respect to the variables x_1 and x_2 (see e.g., [3]). In the more general framework of the quadratic differential operators, this problematic was widely studied (see e.g., [2, 16, 19, 20, 34]) but results were established only for some specific subclasses of these operators. As a byproduct, using interpolation theory, we establish sharp subelliptic estimates that were also conjectured in [19] and widely studied (see e.g., [3, 19, 20, 32]).

Beyond our results, we believe that one of the main interests of this paper consists in the methods we introduce, their possible applications and the links we highlight between the analysis of the properties of semigroups and the study of splitting methods in geometric numerical integration. Our proof relies on a new representation of the polar decomposition of the evolution operators:

$$e^{-tq^w} = e^{-ta_t^w} e^{-itb_t^w}, \tag{1.2}$$

where a_t, b_t are some real valued quadratic forms depending analytically on $0 \leq t \ll 1$, a_t is nonnegative and $e^{-ta_t^w}$ (resp. $e^{-itb_t^w}$) denotes the evolution operator generated by a_t^w (resp. ib_t^w) at time t. The existence of such a representation relies on the *exact classical-quantum correspondence* (through the theory of Fourier Integral Operators developed by L. Hörmander in [21]). This correspondance allows to identify a semigroup generated by the Weyl quantization of a quadratic form with the Hamiltonian flow of this quadratic form (i.e., the exponential of a matrix). The key observation in this paper is that, since $e^{-tib_t^w}$ is unitary, the regularizing effects are entirely driven by $e^{-ta_t^w}$. In other words, a_t encodes all the regularizing effects generated by the nonlinear interactions between the phenomena of diffusions and transports. For example, a formula of Kolmogorov (see e.g., [3]) proves that for the Kolmogorov operator, the factorization (1.2) becomes

$$\forall t \geq 0, \quad e^{t(\partial_{x_2}^2 - x_2\partial_{x_1})} = e^{t(\partial_{x_2} + t\partial_{x_1}/2)^2 + t^3\partial_{x_1}^2/12} e^{-tx_2\partial_{x_1}}.$$

As a consequence, the smoothing properties of this semigroup become as explicit as for the heat equation. Obviously, in general, there is no elementary explicit formula giving a_t. The main technical result of this paper is the derivation of a sharp anisotropic lower bound for a_t in the asymptotic of $0 < t \ll 1$. The starting of this derivation is the observation that a_t^w results from the *backward error analysis* of the Lie splitting method [(1)] associated with the decomposition $2(\operatorname{Re} q)^w = q^w + \bar{q}^w$:

$$e^{-tq^w} e^{-t\bar{q}^w} = e^{-2ta_t^w}.$$

This formula provides a direct way to determine a_t as a function of t and q. Using the generalization [33] of the results of L. Hörmander [23], our results could be extended to non-autonomous equations. Furthermore, in view of [8, 36], we expect that our results could be extended to deal with semigroups generated by inhomogeneous quadratic differential operators. However, these extensions would require some important technicalities. Consequently, they would deserve some further analysis in future works. For the moment, it is not clear how our methods could be extended to deal with non-quadratic operators. It would also deserve some further investigations. We believe that our representation (1.2) could also be useful to analyze some other properties of the semigroups like the propagation of coherent states or singularities. Finally, our methods seem promising to design and analyze rigorously some splitting methods to solve numerically equations of the form (1.1), see [8].

Outline of the work. – Section 2 is devoted to present the main results contained in this paper, put in their bibliographic context and illustrated with examples. In Section 3, we establish the polar decomposition of quadratic semigroups in any positive times whereas Section 4 is devoted to the study of the selfadjoint part for small times. As a byproduct of this decomposition, we study the regularizing effects of semigroups generated by non-selfadjoint quadratic differential operators in Section 5 from which we derive subelliptic estimates enjoyed by accretive quadratic operators in Section 6. Section 7 is an appendix containing the proofs of some technical results.

Convention. – Any complex-valued quadratic form $q : \mathbb{R}^{2n} \to \mathbb{C}$ will be implicitly extended to the complex phase space $\mathbb{C}^{2n}$ in the following way:

$$\forall X \in \mathbb{C}^{2n}, \quad q(X) = X^T Q \overline{X} = q(\operatorname{Re} X) + q(\operatorname{Im} X), \tag{1.3}$$

where $Q \in \mathrm{S}_{2n}(\mathbb{C})$ denotes the matrix of the quadratic form q in the canonical basis of $\mathbb{R}^{2n}$.

Notations. – The following notations will be used all over the work:

1. For all complex matrix $M \in M_n(\mathbb{C})$, M^T denotes the transpose matrix of M while $M^* = \overline{M}^T$ denotes its adjoint.
2. $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{C}^n$ as defined in (2.3).
3. We set $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$ extended to $\mathbb{C}^n$ as explained in the previous convention.

[(1)] This is a classical problematic in geometric numerical integration, we refer the reader to [13] for a presentation of this topic.