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# BLOCH GROUPS, ALGEBRAIC $K$-THEORY, UNITS, AND NAHM'S CONJECTURE

BY Frank CALEGARI, Stavros GAROUFALIDIS AND Don ZAGIER

ABSTRACT. – Given an element of the Bloch group of a number field $F$ and a natural number $n$, we construct an explicit unit in the field $F_n = F(e^{2\pi i/n})$, well-defined up to $n$-th powers of nonzero elements of $F_n$. The construction uses the cyclic quantum dilogarithm, and under the identification of the Bloch group of $F$ with the $K$-group $K_3(F)$ gives (up to an unidentified invertible scalar) a formula for a certain abstract Chern class from $K_3(F)$. The units we define are conjectured to coincide with numbers appearing in the quantum modularity conjecture for the Kashaev invariant of knots (which was the original motivation for our investigation), and also appear in the radial asymptotics of Nahm sums near roots of unity. This latter connection is used to prove Nahm's conjecture relating the modularity of certain $q$-hypergeometric series to the vanishing of the associated elements in the Bloch group of $\overline{\mathbf{Q}}$.

RÉSUMÉ. – Étant donné un élément du groupe de Bloch d'un corps de nombres $F$ et un entier $n$ strictement positif, nous construisons une unité explicite dans l'extension cyclotomique $F_n = F(e^{2\pi i/n})$, bien définie à des puissances $n$-ièmes d'éléments non-nuls de $F_n$ près. La construction utilise le dilogarithme quantique cyclique, et grâce à l'identification du groupe de Bloch de $F$ avec le $K$-groupe $K_3(F)$ donne aussi (à un scalaire inversible non identifié près) une formule pour une certaine classe de Chern abstraite de $K_3(F)$. Les unités que nous définissons coïncident conjecturalement avec les nombres qui apparaissent dans la conjecture de modularité quantique pour l'invariant de Kashaev des nœuds (ce qui constituait la motivation initiale de notre étude), et apparaissent également dans le comportement asymptotique radial des sommes de Nahm au voisinage des racines de l'unité. On utilise cette dernière connexion pour démontrer la conjecture de Nahm qui relie la modularité de certaines séries $q$-hypergéometriques à l'annulation des éléments associés dans le groupe de Bloch de $\overline{\mathbf{Q}}$.

## 1. Introduction

The purpose of the paper is to associate to an element $\xi$ of the Bloch group of a number field $F$ and a primitive $n$-th root of unity $\zeta$ an explicit $S$-unit (where $S$ is independent of $\zeta$ and can often be taken to be trivial) $R_\zeta(\xi)$ in the cyclotomic extension $F_n = F(\zeta)$, well-defined


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up to $n$-th powers of nonzero elements of $F_n$. Our construction uses the cyclic quantum dilogarithm and is shown to agree, up to an unidentified invertible scalar, with the abstract Chern class map on $K_3(F)$ if the latter is identified with the Bloch group. The $S$-unit is also conjectured (and checked numerically in many cases) to coincide with a specific number that appears in the Quantum Modularity Conjecture of the Kashaev invariant of a knot [36]. This was in fact the starting point of our investigation, as described in detail in [11] and in section 1.4 below.

As a surprising consequence of our main theorem we were able to prove a famous conjecture of Werner Nahm asserting that the modularity of certain $q$-hypergeometric series ("Nahm sums") implies the vanishing of certain explicit elements in the Bloch group of $\overline{\mathbf{Q}}$. A precise statement will be given in Section 1.3 of this introduction.

### 1.1. Bloch groups and associated units

We first recall the definition of the classical Bloch group, as introduced by Bloch in [2]. (More precisely, we take the version given by Suslin in [28].) Let $Z(F)$ denote the free abelian group on $F^\times \setminus \{1\}$, i.e. the group of formal finite combinations $\xi = \sum_i n_i[X_i]$ with $n_i \in \mathbf{Z}$ and $X_i \in F^\times \setminus \{1\}$.

Definition 1.1. – The *Bloch group* of a field $F$ is the quotient

$$B(F) = A(F)/C(F), \tag{1}$$

where $A(F)$ is the kernel of the map

$$d : Z(F) \longrightarrow \textstyle\bigwedge^2 F^\times \qquad [X] \mapsto (X) \wedge (1 - X) \tag{2}$$

and $C(F) \subseteq A(F)$ the group generated by the *five-term relation*

$$\xi_{X,Y} = [X] - [Y] + \left[\frac{Y}{X}\right] - \left[\frac{1 - X^{-1}}{1 - Y^{-1}}\right] + \left[\frac{1 - X}{1 - Y}\right] \tag{3}$$

with $X \neq Y$ ranging over all $X \in F^\times \setminus \{1\}$.

We remark that there are a number of different definitions of the Bloch group in the literature which usually agree up to 6-torsion. One harmless modification we may make is to adjoin the elements [0], [1], and [$\infty$] to $B(F)$ subject to the relations:

$$[1] = 0, \quad [\infty] = -[0], \quad [0] = [X] + [1 - X], \ \forall X \in F. \tag{4}$$

We explain in §2 why these new relations don't change $B(F)$. We also discuss an alternative way to define the Bloch group which agrees with Suslin's group up to 2-torsion.

In this paper, we will study an invariant of the Bloch group whose values are units in $F_n$ modulo the $n$-th powers of units, where $n$ is a natural number and $F_n$ the field obtained by adjoining to $F$ a primitive $n$-th root of unity $\zeta = \zeta_n$. The extension $F_n/F$ is Galois with Galois group $G = \mathrm{Gal}(F_n/F)$, and $G$ admits a canonical map

$$\chi : G \longrightarrow (\mathbf{Z}/n\mathbf{Z})^\times \tag{5}$$

determined by $\sigma\zeta = \zeta^{\chi(\sigma)}$. The powers $\chi^j$ ($j \in \mathbf{Z}/n\mathbf{Z}$) of this character define eigenspaces $\left(F_n^\times/F_n^{\times n}\right)^{\chi^j}$ in the obvious way as the set of $x \in F_n^\times/F_n^{\times n}$ such that $\sigma(x) = x^{\chi^j(\sigma)}$ for all $\sigma \in G$, and similarly for $(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n})^{\chi^j}$ or $(\mathcal{O}_{S,n}^\times/\mathcal{O}_{S,n}^{\times n})^{\chi^j}$, where $\mathcal{O}_n$ (resp. $\mathcal{O}_{S,n}$) is the ring of integers (resp. $S$-integers) of $F_n$. Our main result is the following theorem.

THEOREM 1.2. – *Suppose that $F$ does not contain any non-trivial $n$-th root of unity. Then there is a map*

$$R_\zeta : B(F)/nB(F) \longrightarrow \big(\mathcal{O}_{S,n}^\times/\mathcal{O}_{S,n}^{\times n}\big)^{\chi^{-1}} \subset \big(F_n^\times/F_n^{\times n}\big)^{\chi^{-1}} \tag{6}$$

*for some finite set $S$ of primes depending only on $F$. If $n$ is prime to a certain integer $M_F$ depending on $F$, then the map $R_\zeta$ is injective and its image is contained in $\big(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n}\big)^{\chi^{-1}}$, and equal to this if $n$ is prime.*

The map $R_\zeta$ satisfies various natural compatibilities as one varies either $n$ or the field $F$; see Lemmas 2.7 and 2.10.

REMARK 1.3. – Note that the field $F_n$ and the character $\chi$ of (5) depend only on $n$ and not on the primitive $n$-th root of unity $\zeta$. The map $R_\zeta$ from $B(F)$ to $F_n^\times/F_n^{\times n}$ does depend on $\zeta$, but in a very simple way, described by either of the formulas

$$\sigma\big(R_\zeta(\xi)\big) = R_{\sigma(\zeta)}(\xi) \quad (\sigma \in G), \qquad R_\zeta(\xi) = R_{\zeta^k}(\xi)^k \quad (k \in (\mathbf{Z}/n\mathbf{Z})^\times), \tag{7}$$

where the simultaneous validity of these two formulas explains why the image of each map $R_\zeta$ lies in the $\chi^{-1}$ eigenspace of $F_n^\times/F_n^{\times n}$.

REMARK 1.4. – The optimal definition of $M_F$ is somewhat complicated to state. However, one may take it to be $6\,\Delta_F\,|K_2(\mathcal{O}_F)|$. When $n$ is not divisible by 9, one may take $M_F$ to be $2\,\Delta_F\,|K_2(\mathcal{O}_F)|$. (Both assertions are proved in §3.5.)

The detailed construction of the map $R_\zeta$ will be given in Section 2. A rough description is as follows. Let $\xi = \sum n_i[X_i]$ be an element of $Z(F)$ whose image in $\wedge^2(F^\times/F^{\times n})$ under the map induced by $d$ vanishes. We define an algebraic number $P_\zeta(\xi)$ by the formula

$$P_\zeta(\xi) = \prod_i \frac{D_\zeta(x_i)^{n_i}}{D_\zeta(1)^{n_i}}, \tag{8}$$

where $x_i$ is some $n$-th root of $X_i$ and $D_\zeta(x)$ is the *cyclic quantum dilogarithm function*

$$D_\zeta(x) = \prod_{k=1}^{n-1}(1-\zeta^k x)^k. \tag{9}$$

The number $P_\zeta(\xi)$ belongs to the Kummer extension $H_\xi$ of $F$ defined by adjoining all of the $x_i$ to $F_n$ and is well-defined modulo $H_\xi^{\times n}$. We show that for $n$ prime to some $M_F$ it has the form $ab^n$ with $b$ in $H_\xi^\times$ and $a \in F_n^\times$ (or even $a \in \mathcal{O}_n^\times$ under a sufficiently strong coprimality assumption about $n$). Then $R_\zeta(\xi)$ is defined as the image of $a$ modulo $n$-th powers.