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STABILITY CONDITIONS ON KUZNETSOV COMPONENTS

BY Arend BAYER, Martí LAHOZ, Emanuele MACRÌ AND Paolo STELLARI

ABSTRACT. – We introduce a general method to induce Bridgeland stability conditions on semiorthogonal components of triangulated categories. In particular, we prove the existence of Bridgeland stability conditions on the Kuznetsov component of the derived category of Fano three-folds and of cubic fourfolds. As an application, in the appendix, written jointly with Xiaolei Zhao, we give a variant of the proof of the Torelli theorem for cubic fourfolds by Huybrechts and Rennemo.

RÉSUMÉ. – Nous introduisons une méthode générale pour induire des conditions de stabilité de Bridgeland sur des composantes semi-orthogonales des catégories triangulées. En particulier, nous prouvons l'existence de conditions de stabilité de Bridgeland sur la composante de Kuznetsov de la catégorie dérivée de 3-variétés de Fano, ainsi que de 4-variétés cubiques. Comme application, dans l'annexe, écrite conjointement avec Xiaolei Zhao, nous donnons une variante de la preuve de Huybrechts et Rennemo du théorème de Torelli pour les 4-variétés cubiques.

1. Introduction

Main results

Let X be a smooth Fano variety and let $\mathrm{D}^{\mathrm{b}}(X)$ denote its bounded derived category of coherent sheaves. Let $E_1, \dots, E_m \in \mathrm{D}^{\mathrm{b}}(X)$ be an exceptional collection in $\mathrm{D}^{\mathrm{b}}(X)$. We call its right orthogonal complement

$$\begin{aligned}\mathcal{K}u(X) &= \langle E_1, \dots, E_m \rangle^{\perp} \\ &= \left\{ C \in \mathrm{D}^{\mathrm{b}}(X) \,:\, \mathrm{Hom}(E_i, C[p]) = 0,\ \forall i = 1, \dots, m,\ \forall p \in \mathbb{Z} \right\}\end{aligned}$$

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a *Kuznetsov component* of X. In a series of papers, Kuznetsov has shown that much of the geometry of Fano varieties, and their moduli spaces, can be captured efficiently by $\mathcal{K}u(X)$, for appropriate exceptional collections.

Stability conditions on triangulated categories as introduced by Bridgeland in [19] and wall-crossing have turned out to be an extremely powerful tool for the study of moduli spaces of stable sheaves. We connect these two developments with the following two results:

THEOREM 1.1. – *Let X be a Fano threefold of Picard rank* 1 *over an algebraically closed field of characteristic either zero or sufficiently large. Then the Kuznetsov semiorthogonal component $\mathcal{K}u(X)$ has a Bridgeland stability condition.* (1)

The most interesting cases of Theorem 1.1 are Fano threefolds of index two, and those of index one and even genus; in these cases, the theorem holds over any algebraically closed field, independently on the characteristic. We refer to Section 6 for an overview of the classification of Fano threefolds of Picard rank one, and the exceptional collections appearing implicitly in Theorem 1.1.

THEOREM 1.2. – *Let X be a cubic fourfold over an algebraically closed field k with* $\operatorname{char} k \neq 2$. *Then $\mathcal{K}u(X)$ has a Bridgeland stability condition.*

Here $\mathcal{K}u(X)$ is defined by the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle,$$

where H is a hyperplane section. Here $\mathcal{K}u(X)$ is a K3 category (i.e., the double shift [2] is a Serre functor); conjecturally [42, Conjecture 1.1] it is the derived category of a K3 surface if and only if X is rational. Our results also give the first stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ when $\mathcal{K}u(X)$ is not equivalent to the derived category of a twisted K3 surface.

Background and motivation

Wall-crossing for stability conditions on surfaces has had numerous powerful applications, e.g., to the geometry of moduli spaces of stable sheaves [10, 25, 59], or to questions of Brill-Noether type [5, 27, 26]. It is unrealistic to expect similarly systematic results for higher-dimensional varieties: as even Hilbert schemes of curves on $\mathbb{P}^3$ satisfy Murphy's law [80], one should instead expect that wall-crossing lacks any generally effective control. However, Kuznetsov components of Fano varieties are homologically much better behaved than their entire derived category (for example, they can be of Calabi-Yau or Enriques type of smaller dimension). Thus, one can expect that moduli spaces and wall-crossing for objects can be controlled much more effectively, and are thus a natural starting point for extracting geometric results from categorical properties.

The study of Kuznetsov components of derived categories of Fano varieties started with [18], and has seen a lot of recent interest, see e.g., [47, 48, 36] for threefolds, and [42, 2, 1] for the cubic fourfold, as well as [50, 43, 45] for surveys. The interest in them comes from

(1) In some cases, we deduce Theorem 1.1 as an immediate consequence of an explicit description of the Kuznetsov component, which however is stated in the literature only for characteristic zero (and so it also holds if the characteristic of the base field is sufficiently large), see the tables in Section 6.

various directions. They are part of Kuznetsov's powerful framework of Homological Projective Duality [40]. They often seem to encode the most interesting and geometric information about $\mathrm{D}^{\mathrm{b}}(X)$ and moduli spaces of sheaves on X; e.g., several recent constructions of hyperkähler varieties associated to moduli spaces of sheaves on the cubic fourfold are induced by the projection to the K3 category $\mathcal{K}u(X)$ [51] (where moduli spaces naturally come with a holomorphic symplectic structure, due to the fact that $\mathcal{K}u(X)$ is a K3 category). In the case of Fano threefolds, there are a number of unexpected equivalences (some conjectural) between Kuznetsov components of pairs of Fano threefolds of index one and two, see [50] for the theory, and [52] for an application to Hilbert schemes. In the case of cubic fourfolds, as mentioned above, they conjecturally determine rationality of X. Finally, they are naturally related to Torelli type questions: on the one hand, they still encode much of the cohomological information of X; on the other hand, one can hope to recover X from $\mathcal{K}u(X)$ (in some cases when equipped with some additional data); see [15] for such a result for cubic threefolds, and [35] for many hypersurfaces, including cubic fourfolds.

Perhaps the most natural way to extract geometry from $\mathcal{K}u(X)$ is to study moduli spaces of stable objects—hence the interest in the existence of stability conditions on $\mathcal{K}u(X)$. This question was first raised for cubic threefolds in [47], for cubic fourfolds by Addington and Thomas [2] and Huybrechts [32], and in the generality of our results by Kuznetsov in his lecture series [44].

Prior work

When X is a Fano threefold of Picard rank one, stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ have been constructed in [57]. However, in general these do not descend to stability conditions on the semiorthogonal component $\mathcal{K}u(X)$, and due to their importance for moduli spaces, a direct construction of stability conditions on $\mathcal{K}u(X)$ is of independent interest.

For Fano threefolds of index two, our Theorem 1.1 is referring to the decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H)\rangle$. Their deformation type is determined by $d = H^3 \in \{1, 2, 3, 4, 5\}$. The result is straightforward from prior descriptions of $\mathcal{K}u(X)$ for $d \geq 4$ in [71, 18], due to [15] for cubic threefolds ($d = 3$) and new for $d \in \{1, 2\}$. The most interesting cases of index one are those of even genus $g_X = \frac{1}{2}H^3 + 1$, for which Mukai [65] constructed an exceptional rank two vector bundle $\mathcal{E}_2$ of slope $-\frac{1}{2}$; in these cases our theorem refers to the semiorthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{E}_2, \mathcal{O}_X\rangle$. The result is straightforward from previous descriptions of $\mathcal{K}u(X)$ for $g_X \in \{10, 12\}$ in [49, 50], due to [15] for $g_X = 8$, and new for $g_X = 6$.

For cubic fourfolds containing a plane, stability conditions on $\mathcal{K}u(X)$ were constructed in [63], and *Gepner point* stability conditions (invariant, up to rescaling, under the functor (1) in Theorem A.1) in [79]. In this case, $\mathcal{K}u(X)$ is equivalent to the derived category of a K3 surface with a Brauer twist.

Applications

Kuznetsov conjectured an equivalence between $\mathcal{K}u(Y_d)$ and $\mathcal{K}u(X_{4d+2})$ for appropriate pairs Y_d and X_{4d+2}, where Y_d is a Fano threefold of Picard rank one, index two and degree $d \geq 2$, and X_{4d+2} is of index one and genus $2d + 2$ (degree $4d + 2$). Our results may be helpful in reproving known cases, and proving new cases of these equivalences, by