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TORSION CLASSES IN THE EQUIVARIANT CHOW GROUPS OF ALGEBRAIC TORI

BY Federico SCAVIA

Abstract. – We give an example of an algebraic torus T such that the group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is non-trivial. This answers a question of Blinstein and Merkurjev.

Résumé. – On donne un exemple d'un tore algébrique T tel que le groupe $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ est non nul. Ceci répond à une question de Blinstein et Merkurjev.

1. Introduction

Let F be a field, and let G be a linear algebraic group over F. Let $i \geq 0$ be an integer, let V be a linear representation of G over F, and assume that there exists a G-invariant open subscheme U of V such that U is the total space of a G-torsor $U \to U/G$ and $V \setminus U$ has codimension at least $i+1$ in V. Following B. Totaro [10, Definition 1.2], we define

$$\mathrm{CH}^i(BG) := \mathrm{CH}^i(U/G).$$

This definition does not depend on the choice of V and U; see [10, Theorem 1.1]. The graded abelian group $\mathrm{CH}^*(BG) := \bigoplus_{i\geq 0} \mathrm{CH}^i(BG)$ has the structure of a commutative ring with identity.

If T is a split F-torus, and $\hat{T}$ is the character lattice of T, then there is a canonical isomorphism $\mathrm{Sym}(\hat{T}) \simeq \mathrm{CH}^*(BT)$. Thus, if T has rank n, $\mathrm{CH}^*(BT)$ is a polynomial ring with n generators in degree 1, and in particular its underlying additive group is torsion-free.

When G is a finite group, a lot of work on $\mathrm{CH}^*(BG)$ has been carried out by a number of authors, for example N. Yagita [15], P. Guillot [6] and Totaro. Totaro's book [11] is devoted to the study of $\mathrm{CH}^*(BG)$ and to its relation to the group cohomology of G.

When G is a split reductive group, there is an extensive literature dealing with computations of $\mathrm{CH}^*(BG)$. For instance, the ring $\mathrm{CH}^*(BG)$ has been computed for $G = \mathrm{GL}_n$,

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SL_n, Sp_{2n} by Totaro [10], for $G = O_n, SO_{2n+1}$ by Totaro and R. Pandharipande [10] [9], for $G = SO_{2n}$ by R. Field [4], for $G = G_2$ by N. Yagita [14], for $G = PGL_3$ by G. Vezzosi [12], and for $G = PGL_p$ (additively) by A. Vistoli [13].

Let F_s be a separable closure of F, let $\mathcal{G} := \mathrm{Gal}(F_s/F)$ be the absolute Galois group of F. If X is an F-scheme, we define $X_s := X \times_F F_s$. When G is not assumed to be split, a lot less is known about $\mathrm{CH}^*(BG)$. Assume that $G = T$ is an F-torus, not necessarily split. Then we have canonical isomorphisms

$$\mathrm{CH}^1(BT) \simeq \mathrm{CH}^1(BT_s)^{\mathcal{G}} \simeq (\hat{T}_s)^{\mathcal{G}}.$$

The natural homomorphism

$$\mathrm{CH}^2(BT) \to \mathrm{CH}^2(BT_s)^{\mathcal{G}}$$

is not surjective in general; many examples can be obtained from [1, Lemma 4.2, Theorem 4.10, Theorem 4.13].

When X is a smooth variety over F, the natural map

$$\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_s)^{\mathcal{G}}$$

is in general neither injective nor surjective, that is, Galois descent for codimension 2 cycles may fail. It is a difficult and interesting problem to study the kernel and cokernel of the previous map, even for special families of varieties X, and an extensive literature is devoted to it.

Since $\mathrm{CH}^2(BT_s)$ is torsion-free, a norm argument shows that

$$\mathrm{Ker}(\mathrm{CH}^2(BT) \to \mathrm{CH}^2(BT_s)^{\mathcal{G}}) = \mathrm{CH}^2(BT)_{\mathrm{tors}},$$

where $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is the torsion subgroup of $\mathrm{CH}^2(BT)$. The group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ plays a prominent role in work of S. Blinstein and A. Merkurjev, where it appears as the first term of the exact sequence of [1, Theorem B]. In [1, Theorem 4.7], Blinstein and Merkurjev showed that $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is finite and $2 \cdot \mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$. They posed the following question.

QUESTION 1.1 ([1, Question 4.9]). – *Is* $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ *trivial for every torus* T*?*

Merkurjev studied this question further in [8]. He showed that $\mathrm{CH}^2(BT)_{\mathrm{tors}} = 0$ in many cases, for example:

- when BT is 2-retract rational, by [8, Corollary 5.5];
- when the 2-Sylow subgroups of the splitting group of T are cyclic or Klein four-groups, by [8, Proposition 2.1(2), Example 4.3, and Corollary 5.3];
- when char $F = 2$, by [8, Corollary 5.5];
- when $T = R_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ and E/F is a finite Galois extension, by [8, Example 4.2, Corollary 5.3].

The purpose of this paper is to show that Question 1.1 has a negative answer.

THEOREM 1.2. – *There exist a field* F *and an* F*-torus* T *such that* $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ *is not trivial.*

In our example, the splitting group G of T is a 2-Sylow subgroup of the Suzuki group Sz(8), and $F = \mathbb{Q}(V)^G$, where V is a faithful representation of G over $\mathbb{Q}$. The group G has order 64; no counterexample with a splitting group of smaller order can be detected using our method. The torus T has dimension $2^{12} - 2^7 + 1 = (2^6 - 1)^2 = 3969$.

The paper is structured as follows. In Section 2, we recall a construction due to Merkurjev [8], which to every G-lattice L associates an abelian group $\Phi(G, L)$. By a result of Merkurjev, to show that Question 1.1 has a negative answer, it suffices to exhibit G and L such that $\Phi(G, L) \neq 0$; see Theorem 2.3. This reduces Question 1.1 to a problem in integral representation theory. In Section 3, we associate to every finite group G a G-lattice M. In Sections 4 and 5 we show that if the group cohomology of G with $\mathbb{Z}/2$ coefficients satisfies a certain condition, then $\Phi(G, M) \neq 0$; see Proposition 5.3(b). Finally, in Section 6, we show that the condition of Proposition 5.3(b) is satisfied when G is a 2-Sylow subgroup of Sz(8).

2. Merkurjev's reformulation of Question 1.1

Let G be a finite group, and let L be a G-lattice, i.e., a G-module that is finitely generated and free as a $\mathbb{Z}$-module. By definition, the second exterior power $\bigwedge^2(L)$ of L is the quotient of $L \otimes L$ by the subgroup generated by all elements of the form $x \otimes x$, $x \in L$. We denote by $\Gamma^2(L)$ the factor group of $L \otimes L$ by the subgroup generated by $x \otimes y + y \otimes x$, $x, y \in L$. We write $x \wedge y$ for the coset of $x \otimes y$ in $\bigwedge^2(L)$, and $x \star y$ for the coset of $x \otimes y$ in $\Gamma^2(L)$.

We have a short exact sequence

$$0 \to L/2 \xrightarrow{\iota} \Gamma^2(L) \xrightarrow{\pi} \textstyle\bigwedge^2(L) \to 0, \tag{2.1}$$

where $\iota(x + 2L) = x \star x$, and $\pi(x \star y) = x \wedge y$. We write

$$\alpha_L : H^1(G, \textstyle\bigwedge^2(L)) \to H^2(G, L/2)$$

for the connecting homomorphism for (2.1). Recall that a G-lattice is called a permutation lattice if it admits a permutation basis, i.e., a $\mathbb{Z}$-basis stable under the G-action. A G-lattice L' is said to be stably equivalent to L if there exist permutation G-lattices P and P' such that $L \oplus P \simeq L' \oplus P'$.

LEMMA 2.1. – (a) *Assume that L is a permutation G-lattice, and let $x_1, \ldots, x_n$ be a permutation basis of L. Then the homomorphism*

$$\Gamma^2(L) \to L/2, \qquad x_i \star x_j \mapsto 0 \ (i \neq j), \quad x_i \star x_i \mapsto x_i + 2L,$$

defines a splitting of (2.1). Moreover, the homomorphism

$$\textstyle\bigwedge^2(L) \to \Gamma^2(L), \qquad x_i \wedge x_j \mapsto x_i \star x_j \ (i < j)$$

is a section of π.

(b) *Let L' be a G-lattice stably equivalent to L. Then* $\mathrm{Im}(\alpha_L) \simeq \mathrm{Im}(\alpha_{L'})$.

Proof. – This is contained in [8, §2]. □