ISSN 0012-9593

*quatrième série - tome 56 fascicule 2 mars-avril 2023*

# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Pinhas GROSSMAN & Scott MORRISON & David PENNEYS & Emily PETERS & Noah SNYDER

## *The Extended Haagerup fusion categories*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique





**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr


---


**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

**Tarifs**

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.


---



---


ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4ᵉ série, t. 56, 2023, p. 589 à 664

# THE EXTENDED HAAGERUP FUSION CATEGORIES

BY Pinhas GROSSMAN, Scott MORRISON, David PENNEYS, Emily PETERS AND Noah SNYDER

ABSTRACT. – We show there are exactly four fusion categories in the Morita equivalence class of the Extended Haagerup ($\mathcal{EH}$) subfactor, and a unique Morita equivalence between each pair. The $\mathcal{EH}$ subfactor corresponds to the Morita equivalence between $\mathcal{EH}_1$ and $\mathcal{EH}_2$. The new categories $\mathcal{EH}_3$ and $\mathcal{EH}_4$ give new exotic subfactors. The $\mathcal{EH}$ categories are the only known fusion categories unrelated to (quantum) groups or Izumi quadratic categories.

To construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$, we give a general computational recipe to construct fusion categories in the Morita equivalence class of a subfactor. We show that subfactor planar algebra embeddings from $\mathcal{P}_\bullet$ into graph planar algebras are equivalent to pivotal module C* categories over $\mathcal{P}_\bullet$. We construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$ by embedding the $\mathcal{EH}$ planar algebra inside the graph planar algebras of two new graphs. This technique answers a long-standing question of Jones: which graph planar algebras contain a given subfactor planar algebra?

RÉSUMÉ. – Nous montrons qu'il existe exactement quatre catégories de fusion dans la classe d'équivalence au sens de Morita du sous-facteur « Extended Haagerup » ($\mathcal{EH}$), et unicité de l'équivalence entre chaque paire. Le sous-facteur $\mathcal{EH}$ correspond à l'équivalence de Morita entre $\mathcal{EH}_1$ et $\mathcal{EH}_2$. Les nouvelles catégories $\mathcal{EH}_3$ et $\mathcal{EH}_4$ donnent de nouveaux exemples de sous-facteurs exotiques. Les catégories $\mathcal{EH}$ sont les seules catégories de fusion connues qui ne sont pas reliées à un groupe (quantique) ou à une catégorie quadratique d'Izumi.

Pour construire $\mathcal{EH}_3$ et $\mathcal{EH}_4$, nous élaborons une construction générale de catégories de fusion au sein d'une classe d'équivalence de Morita d'un sous-facteur. Nous montrons que les plongements de l'algèbre planaire de sous-facteurs $\mathcal{P}_\bullet$ dans les algèbres planaires de graphe sont en équivalence avec les catégories de modules de pivot C* sur $\mathcal{P}_\bullet$. Nous construisons $\mathcal{EH}_3$ et $\mathcal{EH}_4$ en plongeant l'algèbre planaire $\mathcal{EH}$ dans les algèbres planaires de deux nouveaux graphes. Cette technique répond à une question de Jones de longue date : quelle algèbre planaire de graphe contient une algèbre planaire de sous-facteur donnée?

 doi:10.24033/asens.2541

## 1. Introduction

Group theory provides a unifying language for symmetries across classical mathematics, but in many settings related to quantum mechanics, a more general notion of *quantum symmetry* is required. One of the first appearances of this new kind of symmetry was in the theory of subfactors, i.e., inclusions of von Neumann factors, developed by Jones, Ocneanu, Popa, and others [56, 35, 78, 89, 90]. Here, the appropriate notion of 'Galois theory' requires considering structures more general than groups. But such symmetries have since appeared in many other places, most notably the representation theory of groups of Lie type, polynomial link invariants, topological quantum field theory, conformal field theory, and topological phases of matter. Tensor categories [24, 21] provide the modern language to describe these more general quantum symmetries. Roughly speaking, a tensor category is a category that looks like the category of representations of a group—namely, the category has tensor products and duals. But critically, this tensor product can be noncommutative, so that $X \otimes Y$ and $Y \otimes X$ are not identified, and need not even be isomorphic. The simplest and most widely studied tensor categories are fusion categories, which have strong finiteness and semisimplicity properties analogous to the category of representations of a finite group over a field of characteristic prime to the size of the group.

The most well-known examples of tensor categories come from Lie theory. Following Drinfeld and Jimbo [19, 52], one can deform the universal enveloping algebra of a Lie group, and the category of representations of this quantized universal enveloping algebra is a tensor category. These are not fusion categories, because they are too large. For $\mathrm{SL}_2$, Reshetikhin and Turaev constructed fusion categories built from these quantum groups specialized to roots of unity [92], and this construction was generalized to classical groups by Turaev-Wenzl [100] and all semisimple Lie groups by Andersen and Gelfand-Kazhdan [1, 31]. The big question which motivates this article is whether there are examples of 'exotic' fusion categories which do not 'come from' quantum groups at roots of unity [49]. This is an inherently vague question, because there are many constructions (often of a group-theoretical nature) that can be applied to a fusion category to get a new one. It is possible for a fusion category to look exotic at first, but a later construction might provide a connection to quantum groups. A version of this question was posed by Moore and Seiberg [70] in 1990. Even the simplest special case of this question, whether weakly integral fusion categories come from applying known constructions to the trivial fusion category $\mathsf{Vec}$, remains open [23].

The first 'exotic' examples which appeared to be unrelated to quantum groups came from Haagerup's small index subfactor classification program [44], namely the even parts of the Haagerup and Asaeda-Haagerup subfactors constructed in [2] and the even part of the Extended Haagerup subfactor constructed in [6] (after numerical evidence for existence was given by [50]). However, with time, the first two of these three examples were shown to be related to the more general story of Izumi quadratic fusion categories. [(1)] Izumi generalized the Haagerup subfactor to a possibly infinite family of quadratic $3^G$ subfactors [51, 26]. Recently in [38], Grossman-Izumi-Snyder found all fusion categories Morita equivalent

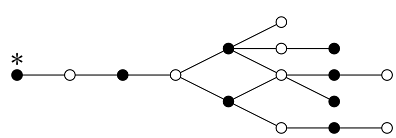
[(1)] It is a very interesting open question whether these quadratic categories can be constructed from quantum groups via Evans-Gannon's conjectural grafting process [26, 37].

to the Asaeda-Haagerup fusion categories, and discovered that one is an Izumi quadratic category. Thus the only known examples of fusion categories which appear unrelated to quantum groups at roots of unity or Izumi quadratic categories are the Extended Haagerup fusion categories.

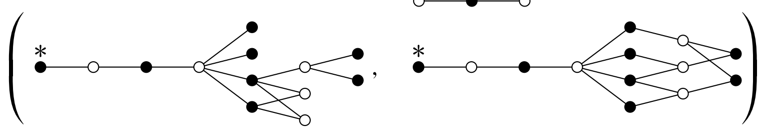
The Extended Haagerup subfactor gives a Morita equivalence between two unitary fusion categories called $\mathcal{EH}_1$ and $\mathcal{EH}_2$. The goal of this paper is to find all fusion categories Morita equivalent to these fusion categories. We find two new fusion categories, seven new subfactors (along with their duals and reduced subfactors—see Section 2), and several interesting new intermediate subfactor lattices. Unlike in the Asaeda-Haagerup case where one of the new categories was a quadratic category, in the Extended Haagerup case neither new category has nontrivial invertible objects and so neither can be a quadratic category. This means the Extended Haagerup fusion categories appear to be more exotic than the Haagerup and Asaeda-Haagerup fusion categories.

THEOREM 1.1. – *There are exactly two further fusion categories in the Morita equivalence class of $\mathcal{EH}_1$ and $\mathcal{EH}_2$, which we call $\mathcal{EH}_3$ and $\mathcal{EH}_4$. Between any two of these four fusion categories, there is exactly one Morita equivalence.*

For every choice of simple object in each of these Morita equivalences, we get a subfactor. In addition to the original 7-supertransitive Extended Haagerup subfactor, we get two new 3-supertransitive subfactors: one is self-dual and comes from the Morita auto-equivalence of $\mathcal{EH}_3$ and the other comes from the Morita equivalence between $\mathcal{EH}_3$ and $\mathcal{EH}_4$. Their principal graphs are:



and



.

The structures of $\mathcal{EH}_3$ and $\mathcal{EH}_4$ are explained in more detail in Section 2. Neither appears to be easily understood using any general techniques, but we encourage the reader to look for a new way to construct them which could give a better understanding of the Extended Haagerup subfactor.

The proof of the main theorem has two parts. On the one hand we need to limit the possible fusion categories and Morita equivalences, and on the other hand we need to construct the remaining possibilities. The former is an application of the techniques introduced in [43], using combinatorial restrictions for compatible fusion rules for the hypothetical fusion categories and bimodule categories.

We construct $\mathcal{EH}_3$ and $\mathcal{EH}_4$ using a general graph planar algebra [57] technique for finding module categories over any fusion category where we have a good skein theoretic description. This technique can be viewed as a generalization of the Ocneanu cell technique for $\mathrm{SU}(n)$ ([79], [27], [87]) to arbitrary tensor categories with good skein theoretic descriptions. From our combinatorial calculation we know that there is at most one module category over each