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DYNAMICAL ALTERNATING GROUPS, STABILITY, PROPERTY GAMMA, AND INNER AMENABILITY

BY DAVID KERR AND ROBIN TUCKER-DROB

ABSTRACT. – We prove that the alternating group of a topologically free action of a countably infinite group Γ on the Cantor set has the property that all of its ℓ^2-Betti numbers vanish and, in the case that Γ is amenable, is stable in the sense of Jones and Schmidt and has property Gamma (and in particular is inner amenable). We show moreover in the realm of amenable Γ that there are many such alternating groups which are simple, finitely generated, and C*-simple. The device for establishing nonisomorphism among these examples is a topological version of Austin's result on the invariance of measure entropy under bounded orbit equivalence.

RÉSUMÉ. – Nous montrons que le groupe alterné d'une action topologiquement libre d'un groupe infini dénombrable Γ sur un espace de Cantor a tous ses nombres de Betti ℓ^2 nuls et, quand Γ est moyennable, est stable au sens de Jones et Schmidt et possède la propriété Gamma (et en particulier est intérieurement moyennable). De plus, dans le cadre de Γ moyennable, nous prouvons qu'il y a de nombreux exemples de tels groupes alternés qui sont simples, de type fini, et C*-simples. L'outil que nous utilisons pour distinguer ces exemples est une version topologique d'un résultat d'Austin sur l'invariance de l'entropie mesurée par équivalence orbitale bornée.

1. Introduction

In the study of von Neumann algebras and their connections to group theory, a pivotal role is played by the II_1 factors, i.e., the infinite-dimensional von Neumann algebras having trivial center and a (necessarily faithful and unique) normal tracial state. A II_1 factor is said to have *property Gamma* if it has an asymptotically central net of unitaries with trace zero. This concept was introduced by Murray and von Neumann in order to show that there exists a II_1 factor, in this case the group von Neumann algebra $\mathscr{L}F_2$ of the free group on two generators, which is not isomorphic to the hyperfinite II_1 factor [40]. While the stronger property of injectivity was later identified as the von Neumann algebra analogue of amenability for groups, and indeed as being equivalent to this amenability in the case of group von Neumann algebras, Murray and von Neumann's argument nevertheless hinted at

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a connection to amenability through its use of almost invariant measures and paradoxicality in a way that, as they pointed out, paralleled Hausdorff's paradoxical decomposition of the sphere. In fact property Gamma has come, perhaps surprisingly, to play a significant role in the study of operator-algebraic amenability. It appears as a technical tool in Connes's celebrated work on the classification of injective factors [9], and its C*-algebraic analogue has very recently been used to establish the equivalence of finite nuclear and $\mathscr{Z}$-stability in the Toms-Winter conjecture [6]. The explicit connection to group-theoretic amenability was made by Effros [18], who showed that, for an ICC discrete group G, if the group von Neumann algebra $\mathscr{L}G$ (which in this case is a II_1 factor) has property Gamma then G is *inner amenable*, meaning that there exists an atomless finitely additive probability measure on $G \setminus \{1_G\}$ which is invariant under the action of G by conjugation. The converse of this implication turns out to be false, however, as was shown by Vaes in [47].

Let us say for brevity that an ICC discrete group G has *property Gamma* if $\mathscr{L}G$ has property Gamma. When such a group is nonamenable it exhibits the kind of behavior that brings it under the compass of Popa's deformation/rigidity program. Indeed Peterson and Sinclair showed in [43] that if G is countable and nonamenable and has property Gamma then each of its weakly mixing Gaussian actions, including its Bernoulli action over a standard atomless base, is $\mathscr{U}_{\mathrm{fin}}$-cocycle superrigid, i.e., every cocycle taking values in a Polish group that embeds as a closed subgroup of the unitary group of some II_1 factor, and in particular every cocycle into a countable discrete group, is cohomologous to a homomorphism. This implies that if G has no nontrivial finite normal subgroups then each of its weakly mixing Gaussian actions is orbit equivalence superrigid, i.e., any orbit equivalence with another ergodic p.m.p. action of a countable discrete group implies that the actions are conjugate modulo an isomorphism of the groups. In [46] Ioana and the second author strengthened Peterson and Sinclair's result by showing that the conclusion holds more generally if property Gamma is replaced by inner amenability. In Corollary D of [8] it was shown that inner amenability implies the vanishing of the first ℓ^2-Betti number, which is a necessary condition for $\mathscr{U}_{\mathrm{fin}}$-cocycle superrigidity (Corollary 1.2 of [43]) and hence can also be derived from [46].

Inner amenability is also related to Jones and Schmidt's notion of stability for p.m.p. equivalence relations [24], which is an analogue of the McDuff property for II_1 factors. We say that an ergodic p.m.p. equivalence relation is *JS-stable* if it is isomorphic to its product with the unique ergodic hyperfinite p.m.p. equivalence relation. A countable group is said to be *JS-stable* if it admits a free ergodic p.m.p. action whose orbit equivalence relation is JS-stable. Just as McDuff implies property Gamma in the II_1 factor setting, JS-stability for a group implies inner amenability (Proposition 4.1 of [24]). However, there exist ICC groups with property Gamma which are not JS-stable, as shown by Kida [30]. Moreover, while JS-stability for an action implies that the crossed product is McDuff, JS-stability for a group does not imply that the group has property Gamma [31]. We refer the reader to [14] for a more extensive discussion of the relationships between all of these properties.

In this paper we show that actions of amenable groups on the Cantor set provide a rich source of groups which have property Gamma and are JS-stable, and that many of these groups are simple, finitely generated, and C*-simple (C*-simplicity means that the reduced group C*-algebra is simple, or equivalently that there are no nontrivial amenable uniformly recurrent subgroups [28], and is a strengthening of nonamenability).

As mentioned above, inner amenability implies the vanishing of the first ℓ^2-Betti number. It is also known that JS-stability implies the vanishing of all ℓ^2-Betti numbers, for in this case the group is measure equivalent to a product of the form $H \times \mathbb{Z}$ for some group H, which has the property that all of its ℓ^2-Betti numbers vanish [7], and the vanishing of all ℓ^2-Betti numbers is an invariant of measure equivalence [20]. We will begin by proving in Theorem 3.2 that for every action $\Gamma \curvearrowright X$ of a countably infinite group (amenable or not) on the Cantor set, every infinite subgroup of the topological full group containing the alternating group has the property that all of its ℓ^2-Betti numbers vanish. We then show in Theorem 4.8 that, for an action $\Gamma \curvearrowright X$ of a countable amenable group on the Cantor set, every such subgroup of the topological full group is JS-stable. To carry this out we use the stability sequence criterion for JS-stability due to Jones and Schmidt in the free ergodic case [24] and to Kida more generally [32] along with an invariant random subgroup generalization of a result from [46] that yields a stability sequence from the existence of fiberwise stability sequences in an equivariant measure disintegration of an action whose orbit equivalence relation is hyperfinite (Theorem 4.4).

In Theorem 5.5 we establish property Gamma for the topological full group, and every subgroup thereof containing the alternating group $\mathsf{A}(\Gamma, X)$, of a topologically free action $\Gamma \curvearrowright X$ of a countable amenable group on the Cantor set (the ICC condition is automatic in this case by Proposition 5.1). As a corollary these subgroups of the topological full group are inner amenable, and in Section 6 we provide two direct proofs of this inner amenability whose techniques may be of use in other contexts. One of these proofs does not require topological freeness and yields inner amenability merely assuming that the subgroup is infinite, which is automatic if $\mathsf{A}(\Gamma, X)$ is nontrivial.

Perhaps most interesting is the situation when the action $\Gamma \curvearrowright X$ is minimal and expansive, as the first of these two conditions implies that the alternating group is simple and the second implies, under the assumption that Γ is finitely generated, that the alternating group finitely generated [41]. Under these additional hypotheses it is possible for the alternating group not to be amenable, as Elek and Monod constructed in [19] a free minimal expansive $\mathbb{Z}^2$-action whose topological full group contains a copy of F_2, and in this case the alternating group coincides with the commutator subgroup of the topological full group and hence is nonamenable. We thereby obtain an example of a simple finitely generated nonamenable group with property Gamma, and in particular an example of a simple finitely generated nonamenable inner amenable group, which answers a question that was posed to the second author by Olshanskii. It has also recently been shown by Szőke that every infinite finitely generated group which is not virtually cyclic admits a free minimal expansive action on the Cantor set whose topological full group contains F_2 [45].

We push this line of inquiry further by showing that many finitely generated torsion-free amenable groups, including those that are residually finite and possess a nontorsion element with infinite conjugacy class, admit an uncountable family of topologically free minimal actions on the Cantor set whose alternating groups are C*-simple (and in particular nonamenable) and pairwise nonisomorphic (Theorem 8.9). These actions are constructed so as to have different values of topological entropy, so that the pairwise nonisomorphism follows by combining the following two facts: