ISSN 0012-9593

quatrième série - tome 56 fascicule 1 janvier-février 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Lingrui GE & Jiangong YOU & Qi ZHOU

Exponential Dynamical Localization: Criterion and Applications

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 56, 2023, p. 91 à 126

EXPONENTIAL DYNAMICAL LOCALIZATION: CRITERION AND APPLICATIONS

BY LINGRUI GE, JIANGONG YOU AND QI ZHOU

ABSTRACT. – A new criterion for exponential dynamical localization in expectation (EDL) is given for ergodic families of operators acting on $\ell^2(\mathbb{Z}^d)$. As applications, the EDL for a class of quasi-periodic long-range operators on $\ell^2(\mathbb{Z}^d)$ is proven by the criterion and quantitative almost reducibility.

RÉSUMÉ. – On donne un nouveau critère pour la localisation dynamique exponentielle (LDE) pour des familles ergodiques d'opérateurs agissants sur $\ell^2(\mathbb{Z}^d)$. Comme applications, on montre la LDE pour une classe d'opérateurs à longue portée quasi-périodique sur $\ell^2(\mathbb{Z}^d)$, par le critère et la presque réductibilité quantitative.

1. Introduction

Localization of particles and waves in disordered media is one of the most intriguing phenomena in solid-state physics. Dating back to 1958, Anderson [5] firstly used a tight-binding model of an electron in a disordered lattice to explain the physical phenomenon that anomalous long relaxation time of electron spins in doped semiconductors. Since that ground breaking work, physicists found Anderson localization has strong connection with integer quantum Hall effect [2, 35, 54], which also plays an important role in the emerging subject of optical crystals [27], e.t.c. One may consult [51] for more about the history of the localization theory.

The mathematical models of the above problems appear often as ergodic families of Schrödinger operators. Let $(\Omega, d\mu, S)$ be a measurable space where $d\mu$ is a Borel probability measure, $S = \{S_n, n \in \mathbb{Z}^d\}$ is an ergodic family of maps from Ω to Ω such that $S_{m+n} = S_m S_n$. A measurable family $(H_\omega)_{\omega\in\Omega}$ of bounded linear self-adjoint operators on $\ell^2(\mathbb{Z}^d)$ is called ergodic if $H_{S_n\omega} = T_n H_\omega T_{-n}$ where T_n is the translation in $\mathbb{Z}^d$ by the vector n. Motivated by physical backgrounds, localization property of ergodic families of operators $(H_\omega)_{\omega\in\Omega}$ has been extensively studied in the past sixty years. There are several different mathematical definitions for localization from weak sense to strong sense. The

 doi:10.24033/asens.2529

weakest one is the Anderson localization (AL): H_ω is said to display Anderson localization for $a.e.\ \omega$ if H_ω has pure point spectrum with exponentially decaying eigenfunctions for $a.e.\ \omega$. However, in physics, localization often means dynamical localization (DL), i.e., the wave-packets under the Schrödinger time evolution e^{itH_ω} keep localized if the initial wave-packet is localized. One generally accepted definition of DL is the following: for any $q > 0$,

$$\sup_t \sum_{n\in\mathbb{Z}} (1+|n|)^q |\langle \delta_n, e^{-itH_\omega}\delta_0\rangle| < C(\omega) < \infty. \tag{1.1}$$

When considering an ergodic family operators $(H_\omega)_{\omega\in\Omega}$, the above defined DL often holds only for a full measure of ω, moreover $C(\omega)$ is not uniform in ω. So it is natural to consider the dynamical localization in expectation. The strongest dynamical localization is the *exponential dynamical localization in expectation* (EDL) [2, 40, 41]:

$$\int_\Omega \sup_{t\in\mathbb{R}} |\langle \delta_k, e^{-itH_\omega}\delta_\ell\rangle| d\mu \le Ce^{-\gamma|k-\ell|}, \tag{1.2}$$

where

$$\gamma(H) := \liminf_{n\to\infty}\Big(-\frac{1}{|n|}\ln \int_\Omega \sup_{t\in\mathbb{R}} |\langle \delta_n, e^{-itH_\omega}\delta_0\rangle| d\mu\Big)$$

is defined as the *exponential decay rate in expectation* [40, 41]. This quantity is useful since it is obviously connected to the minimal inverse correlation length. As pointed out in [2, 40], EDL leads to various interesting physical conclusions, for example, the exponential decay of the two-point function at the ground state and positive temperatures with correlation length staying uniformly bounded as temperature goes to zero. Moreover, it is indeed EDL that is often implicitly assumed as manifesting localization in physics literature. That makes it particularly interesting to establish EDL for physically relevant models.

Dynamical localization has already been well studied for the random cases [1, 3, 4, 23, 50]. So we will focus on the quasi-periodic models, which even have stronger backgrounds in physics [14, 55]. As pointed out in [32], establishing dynamical localization in the quasi-periodic case requires approaches that are quite different from that of the random case (which usually uses Aizenman-Molchanov's fractional moments method [3]). So far, there are few EDL results for one dimensional quasi-periodic Schrödinger operators and there is no EDL result for higher dimensional quasi-periodic Schrödinger operators. In this paper, we will provide a general criterion for EDL by information of eigenfunctions and apply it to some popular quasi-periodic models. More applications to other quasiperiodic models will be given in our forthcoming paper. Hopefully, the criterion can also be used to deal with other ergodic Schrödinger operators.

Dating back to 1980s, the most extensively studied quasiperiodic models have the following form:

$$(L_{V,\lambda W,\alpha,\theta}u)_n = \sum_{k\in\mathbb{Z}^d} V_k u_{n-k} + \lambda W(\theta + \langle n,\alpha\rangle)u_n, \tag{1.3}$$

where $\alpha \in \mathbb{T}^d$ is the frequency, $V_k \in \mathbb{C}$ is the k-th Fourier coefficient of a real analytic function $V : \mathbb{T}^d \to \mathbb{R}$, W is a continuous function defined on $\mathbb{T}$. It is an ergodic self-adjoint operator defined on $\ell^2(\mathbb{Z}^d)$. There are two fundamental results when considering the localization property of $L_{V,\lambda W,\alpha,\theta}$. Inspired by the pioneering work of Fröhlich-Spencer-Wittwer [29] and Sinai [60], Chulaevsky-Dinaburg [22] proved that if W is a cosine-like function, then for any fixed phase θ, and for positive measure α, $L_{V,\lambda W,\alpha,\theta}$ has AL for sufficiently large coupling constant λ. Bourgain [15] generalized this result to the case of arbitrary real analytic function W.

If $W(\theta) = 2\cos(2\pi\theta)$, then (1.3) reduces to the famous quasiperiodic long-range operator:

$$(L_{V,\lambda,\alpha,\theta}u)_n = \sum_{k\in\mathbb{Z}^d} V_k u_{n-k} + 2\lambda\cos(\theta + \langle n,\alpha\rangle)u_n. \tag{1.4}$$

The operator (1.4) has received a lot of attention [10, 19, 31, 30] since it is the Aubry duality of the general quasiperiodic Schrödinger operator defined on the one dimensional lattice

$$(H_{\lambda^{-1}V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda^{-1}V(\theta_1 + n\alpha_1, \dots, \theta_d + n\alpha_d)u_n, \tag{1.5}$$

and thus (1.4) carries amounts of information on (1.5) [21, 33].

If furthermore $V(\theta) = 2\cos(2\pi\theta)$, then (1.4) furthermore reduces to the most famous almost Mathieu operator (AMO):

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(\theta + n\alpha)u_n, \tag{1.6}$$

The AMO was first introduced by Peierls [56], as a model for an electron on a 2D lattice, acted on by a homogeneous magnetic field [36, 58]. It plays the central role in the Thouless et al theory of the integer quantum Hall effect [61]. This model has been extensively studied not only because of its importance in physics [14, 55], but also as a fascinating mathematical object. More detailed properties of AMO will be discussed separately in Subsection 1.3.

1.1. EDL for quasi-periodic long-range operators

Considering (1.4), if $d = 1$, Bourgain-Jitomirskaya [19] proved that for any fixed Diophantine number α [(1)], $L_{V,\lambda,\alpha,\theta}$ has DL for sufficiently large λ and a.e. θ. This result is non-perturbative in the sense that the largeness of λ doesn't depend on the Diophantine constant of α.

If $d \geq 2$, Jitomirskaya and Kachkovskiy [39] proved that for fixed $\alpha \in DC_d$, $L_{V,\lambda,\alpha,\theta}$ has pure point spectrum for large enough λ and $a.e.\ \theta$. Compared to the result of dimension one, here one can only obtain the perturbative result, i.e., the largeness of λ depends on α.

We remark that Jitomirskaya-Kachkovskiy's results [39] are based on Eliasson's full measure reducibility results [25]. By Aubry duality, although all the eigenfunctions decay exponentially, they don't have uniform decay rate. That's the reason why they could only obtain pure point spectrum instead of AL. What we will prove is, under the exact same setting as in [39], the family of operator $(L_{V,\lambda,\alpha,\theta})_{\theta\in\mathbb{T}}$ not only displays AL but also EDL. Now we precisely formulate our result.

(1) The number $\alpha \in \mathbb{T}^d$ is called *Diophantine*, denoted by $\alpha \in \mathrm{DC}_d(\kappa',\tau)$, if there exist $\kappa' > 0$ and $\tau > d-1$ such that

$$\mathrm{DC}_d(\kappa',\tau) := \left\{\alpha \in \mathbb{T}^d : \inf_{j\in\mathbb{Z}} |\langle n,\alpha\rangle - j| > \frac{\kappa'}{|n|^\tau}, \quad \forall\, n \in \mathbb{Z}^d\backslash\{0\}\right\}. \tag{1.7}$$

Let $\mathrm{DC}_d := \bigcup_{\kappa'>0,\,\tau>d-1} \mathrm{DC}_d(\kappa',\tau)$.