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CONVERGENCE OF PERCOLATION ON UNIFORM QUADRANGULATIONS WITH BOUNDARY TO SLE_6 ON $\sqrt{8/3}$-LIOUVILLE QUANTUM GRAVITY

by

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Abstract. — Let Q be a free Boltzmann quadrangulation with simple boundary decorated by a critical ($p = 3/4$) face percolation configuration. We prove that the chordal percolation exploration path on Q between two marked boundary edges converges in the scaling limit to chordal SLE_6 on an independent $\sqrt{8/3}$-Liouville quantum gravity disk (equivalently, a Brownian disk). The topology of convergence is the Gromov-Hausdorff-Prokhorov-uniform topology, the natural analog of the Gromov-Hausdorff topology for curve-decorated metric measure spaces. We also obtain analogous scaling limit results for face percolation on the uniform infinite half-plane quadrangulation with simple boundary, and for site percolation on a uniform triangulation with simple boundary. Our method of proof is robust and, up to certain technical steps, extends to any percolation model on a random planar map which can be explored via peeling.

Résumé **(Convergence de la percolation sur des quadrangulations uniformes avec bord vers le SLE_6 sur la gravité quantique de Liouville de paramètre $\sqrt{8/3}$)**

Soit une quadrangulation de Boltzmann avec bord simple décorée par une percolation critique ($p = 3/4$) sur ses faces. Nous montrons que le processus d'exploration de cette percolation entre deux arêtes marquées du bord converge dans la limite d'échelle vers un processus SLE_6 chordal dessiné sur un disque quantique au sens de la gravité quantique de Liouville de paramètre $\sqrt{8/3}$ (i.e., un disque Brownien). La topologie considérée ici est la topologie de Gromov-Hausdorff-Prokhorov uniforme, qui est l'analogue de la topologie de Gromov-Hausdorff pour l'étude des espaces métriques mesurés décorés par une courbe. Nous obtenons également des résultats similaires pour la percolation critique sur les faces de quadrangulations uniformes infinies du demi-plan, ainsi que pour la percolation critique par sites sur une triangulation uniforme avec bord. La méthode de preuve est robuste et, modulo quelques ajustements techniques, s'étend essentiellement à tous les modèles de percolation sur les cartes planaires qui peuvent être explorés par épluchage.

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1. Introduction

1.1. Overview. — In the past several decades, a vast literature concerning statistical mechanics models in two dimensions has been developed. This work includes models on on deterministic lattices (such as $\mathbb{Z}^2$) as well as on random planar maps, i.e., random graphs embedded in the plane, viewed modulo orientation-preserving homeomorphisms. A central focus in this field is to show that these statistical mechanics models converge, under an appropriate scaling limit, to continuum models.

In the case of critical models on deterministic lattices, the limiting objects are often described (or conjectured to be described) in terms of *Schramm-Loewner evolution (SLE)* [**69**], a one-parameter family of random fractal curves; see, e.g., [**51, 76, 77, 70**]. SLE has been conjectured to arise in this context because in his original derivation [**69**] Schramm showed that it is characterized by the fact that it is *conformally invariant* and satisfies a spatial Markov property called the *domain Markov property*; these two properties together are sometimes referred to as the *conformal Markov property*. Many discrete models in two dimensions satisfy an exact spatial Markov property and are conjectured to be conformally invariant in the limit and therefore be SLEs. For critical models on random planar maps, one instead gets SLE curves in a random geometry which arises as the scaling limit of the underlying random planar map. This random geometry can be described in terms of *Liouville quantum gravity (LQG)*, a one-parameter family of random fractal surfaces. LQG surfaces with parameter $\gamma = \sqrt{8/3}$ are especially important since such surfaces describe the scaling limits of *uniform* random planar maps, i.e., planar maps where each possibility is assigned equal probability. Certain special $\sqrt{8/3}$-LQG surfaces are equivalent, as metric measure spaces, to *Brownian surfaces*, such as the Brownian map [**53, 57**] or the Brownian disk [**13**].

The goal of this article is to show that a certain statistical mechanics model—namely, critical percolation—on certain types of random planar maps converges to SLE_6 on a $\sqrt{8/3}$-LQG surface, or equivalently a Brownian surface. The topology of convergence is the so-called *Gromov-Hausdorff-Prokhorov uniform topology*, the natural analog of the Gromov-Hausdorff topology for curve-decorated metric measure spaces. We will provide more background about our result and the relevant mathematical objects shortly, but before we do so let us first comment briefly on our proof strategy.

The proof of our main scaling limit result has three main steps.

1. Show that percolation on a random planar map is tight with respect to the above topology.
2. Show that the desired limiting object—namely SLE_6 on a $\sqrt{8/3}$-LQG surface—is uniquely characterized by a certain set of properties (essentially, the topology of the curve plus a "LQG" variant of the domain Markov property).
3. Show that every possible subsequential limit of our discrete objects satisfies the properties in this characterization theorem.

This proof outline is quite different from known scaling limit proofs for models on deterministic lattices toward SLE, which typically show directly that the Loewner driving function of the discrete curve converges to a multiple of Brownian motion (as opposed to using Schramm's conformal Markov characterization of SLE). Our argument is also very different from the proof of the convergence of self-avoiding walk on random planar maps to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG, with respect to the same topology we consider here, from **[37]**.

In this paper, we will carry out steps 1 and 3, which both involve purely discrete (random planar map) arguments. Step 2 is carried out in the companion paper **[38]**, using purely continuum (SLE/LQG) arguments which are of quite a different flavor and make use of a different mathematical toolbox in comparison to this paper. We review all of the SLE/LQG results which are needed for the proofs of our main results, including the characterization theorem from **[38]**, in Section 2.3 below. We note that in the course of proving this characterization theorem, **[38]** also establishes characterizations for other variants of SLE_κ curves on γ-LQG surfaces for $\gamma \in (\sqrt{2}, 2)$ and $\kappa = 16/\gamma^2 \in (4, 8)$, which may have applications to proving other scaling limit results for statistical mechanics models in random geometries.

1.1.1. *Percolation.* — Let G be a graph and $p \in [0, 1]$. Recall that *site (resp. bond) percolation* on G with parameter $p \in [0, 1]$ is the model in which each vertex (resp. edge) of G is declared to be open independently with probability p. A vertex (resp. edge) which is not open is called closed. If G is a planar map (i.e., a graph together with an embedding into the plane so that no two edges cross), one can also consider *face percolation*, equivalently site percolation on the dual map, whereby each face is open with probability p and closed with probability $1 - p$. We refer to **[30, 15]** for general background on percolation.

Suppose now that G is an infinite graph with a marked vertex v. The first question that one is led to ask about percolation on G, which was posed in **[17]**, is whether there exists an infinite *open cluster* containing v, i.e., a connected set of open vertices, edges, or faces (depending on the choice of model). For $p \in [0, 1]$, let $\phi(p)$ be the probability that there is such an open cluster containing v and let $p_c = \sup\{p \in [0, 1] : \phi(p) = 0\}$ be the *critical probability* above (resp. below) which there is a positive (resp. zero) chance there is an infinite open cluster containing v. The value of p_c is in general challenging to determine, but has been identified in some special cases. For example, it is known that $p_c = 1/2$ for both bond percolation on $\mathbb{Z}^2$ and for site percolation on the triangular lattice **[48]**. As we will explain below, p_c has also been identified for a number of random planar map models.

The next natural question that one is led to ask is whether the percolation configuration at criticality ($p = p_c$) possesses a *scaling limit*, and this is the question in which we will be interested in the present work. For percolation on a two-dimensional lattice when $p = p_c$, the interfaces between open and closed clusters are expected to converge in the scaling limit to Schramm-Loewner evolution (SLE)-type curves **[69]** with parameter $\kappa = 6$. The reason for this is that the scaling limits of these percolation interfaces are conjectured to be conformally invariant (attributed to Aizenman by Langlands, Pouliot, and Saint-Aubin in [**49**]) with crossing probabilities which satisfy Cardy's formula [**21**]. The particular value $\kappa = 6$ is obtained since this is the only value for which SLE possesses the *locality property* [**50**], which is a continuum analog of the statement that the behavior of a percolation interface is not affected by the percolation configuration outside of a sub-graph of the underlying lattice until it exits that sub-graph. This conjecture has been proven in the special case of site percolation on the triangular lattice by Smirnov [**76**]; see [**18**] for a detailed proof of the scaling limit result and [**43**] for a proof of convergence in the so-called natural parameterization. The proof of [**76**] relies crucially on the combinatorics of site percolation on the triangular lattice and does not generalize to other percolation models.

In this paper we will prove scaling limit results for percolation on *random planar maps* and identify the limit with SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity, equivalently, SLE_6 on a Brownian surface. Statistical mechanics models on random planar maps and deterministic lattices are both of fundamental importance in mathematical physics. Indeed, both are well-motivated in the physics literature and both possess a rich mathematical structure. Many questions (e.g., scaling limit results for random curves toward SLE) can be asked for both random planar maps and deterministic lattices, and it is not in general clear which setting is easier. There are scaling limit results which have been proven for models on deterministic lattices but not random planar maps (e.g., the convergence of Ising model interfaces to SLE_3 [**77**] or, prior to this paper, the convergence of percolation to SLE_6) or for random planar maps but not deterministic lattices (e.g., the convergence of self-avoiding walk to $\mathrm{SLE}_{8/3}$ [**37**] or peanosphere scaling limit results [**25, 75, 47, 32**]).

We will focus on the particular model of face percolation on a random quadrangulation. (We will discuss the universality of the scaling limit in Section 8 in detail in the setting of site percolation on triangulations.) Critical probabilities for several percolation models on random planar maps are computed in [**4**], building on ideas of [**2, 3**]; in particular, $p_c = 3/4$ for face percolation on random quadrangulations. The fact that $p_c = 3/4$ and not $1/2$ is related to the asymmetry between open and closed faces: open faces are considered adjacent if they share an edge, whereas closed faces are considered adjacent if they share a vertex. See [**68, 56**] for the computation of p_c for other planar map models.

One useful feature of percolation on random planar maps is the so-called *peeling procedure* which allows one to describe the conditional law of the remaining map when we explore a single face. For face percolation with open/closed boundary conditions, the peeling process gives rise to a natural path from the root edge to the target edge which we call the *percolation exploration path* (see Section 1.2.2 for a precise definition of this path). The peeling exploration path is closely related to, but not in general identical to, the percolation interface from the root edge to the target edge; see Section 3.4 for further discussion of this relationship. In the special case of site percolation on a triangulation the percolation exploration path is the same as the percolation interface.

1.1.2. *Limiting object: SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity.* — For $\gamma \in (0,2)$, a *γ-Liouville quantum gravity (LQG) surface* is (formally) the random surface parameterized by a domain $D \subset \mathbb{C}$ whose Riemannian metric tensor is $e^{\gamma h(z)}\, dx \otimes dy$, where h is some variant of the Gaussian free field (GFF) on D and $dx \otimes dy$ is the Euclidean metric tensor. This does not make rigorous sense since h is a distribution, not a function. However, it was shown in **[26]** that one can make rigorous sense of the volume form associated with a γ-LQG surface, i.e., one can define a random measure μ_h on D which is a limit of regularized versions of $e^{\gamma h(z)}\, dz$ where dz is the Euclidean volume form (see **[67]** and the references therein for a more general approach to constructing measures of this form). Hence a γ-LQG surface can be viewed as a random measure space together with a conformal structure.

In the special case when $\gamma = \sqrt{8/3}$, it is shown in **[62, 65, 64]**, building on **[59, 63, 61]**, that (D, h) can also be viewed as a random metric space, i.e., one can construct a metric $\mathfrak{d}_h$ on D which is interpreted as the distance function associated with $e^{\gamma h(z)}\, dx \otimes dy$. For certain special $\sqrt{8/3}$-LQG surfaces introduced in **[25, 74]**, the metric measure space structure of a $\sqrt{8/3}$-LQG surface is equivalent to a corresponding Brownian surface. In particular, the Brownian map, the scaling limit of the uniform quadrangulation of the sphere **[53, 57]**, is equivalent to the *quantum sphere*. Also, the Brownian half-plane, the scaling limit of the uniform quadrangulation of the upper-half plane $\mathbb{H}$ in the Gromov-Hausdorff topology **[33, 6]**, is equivalent to the $\sqrt{8/3}$-*quantum wedge*. Finally, the Brownian disk, the scaling limit of the uniform quadrangulation of the disk $\mathbb{D}$ **[13]**, is equivalent to the *quantum disk*.

The metric measure space structure of a $\sqrt{8/3}$-LQG surface a.s. determines the conformal structure **[64]**, so we have a canonical way of embedding a Brownian surface into $\mathbb{C}$. This enables us to define an independent SLE_6 on the Brownian map, half-plane, and disk as a curve-decorated metric measure space by first embedding the Brownian surface into $\mathbb{C}$ to get a $\sqrt{8/3}$-LQG surface and then sampling an independent SLE_6 connecting two marked points. The canonical choice of parameterization is the so-called *quantum natural time* with respect to this $\sqrt{8/3}$-LQG surface, a notion of time which is intrinsic to the curve decorated quantum surface **[25]**. See Section 2.3 for more on $\sqrt{8/3}$-LQG surfaces and their relationship to SLE_6 and to Brownian surfaces.

In the companion paper **[38]**, we prove a characterization of SLE_6 on a Brownian surface by a set of simple properties, which is re-stated as Theorem 2.7. This characterization is in some ways similar to Schramm's **[69]** characterization of SLE in terms of conformal Markov property, in that it involves a continuum analog of the Markov property for percolation interfaces on a random planar map. However, the properties in our characterization theorem are different than those in Schramm's characterization, and the proof is extremely different. As discussed above, this characterization plays a fundamental role in the proof of our main results.