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# FIXED POINTS OF NILPOTENT ACTIONS ON $\mathbb{R}^2$

*by*

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*En hommage à Jean-Christophe Yoccoz*

***Abstract.*** — We show several results providing global fixed points for nilpotent groups of orientation-preserving $C^1$ diffeomorphisms of the plane $\mathbb{R}^2$. The main cases are namely groups of diffeomorphisms of the sphere such that $\infty$ is a global fixed point, groups of diffeomorphisms preserving a non-empty compact set and finally groups of diffeomorphisms preserving a probability measure.

***Résumé* (Points fixes des actions nilpotentes de $\mathbb{R}^2$).** — Nous montrons plusieurs résultats d'existence de points fixes pour des groupes nilpotents de difféomorphismes de classe $C^1$ du plan $\mathbb{R}^2$. Les cas principaux sont ceux de groupes de dfféomorphismes de la sphère fixant le point à l'infini, de groupes de difféomorphismes fixant un compact donné du plan, et finalement de groupes de difféomorphismes préservant une mesure de probabilité.

## 1. Introduction

We present here some results of existence of a global fixed point for a nilpotent group $G$ of orientation preserving plane diffeomorphisms, which means a point fixed by every element of $G$. We denote $\mathrm{Diff}^1_+(\mathbb{R}^2)$ the group of orientation preserving diffeomorphisms of class $C^1$ of $\mathbb{R}^2$.

Let us state the key result:

***Theorem 1.*** — *Let $G$ be a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that preserves a non-empty compact set. Then $G$ has a global fixed point.*

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A global fixed point is a common fixed point for all elements of the group $G$. Theorem 1 was proved by Franks, Handel and Parwani **[6]** in the particular case of a finitely generated abelian group and extended by the third author **[10]** to the case of a finitely generated nilpotent group. We will see here that the finiteness condition is not necessary.

The group $G$ being nilpotent, it preserves a Borel probability measure (which means that the measure is invariant by every element of $G$) if it preserves a non-empty compact set. Does the conclusion of the theorem still holds supposing this weaker condition? We will see that the answer is yes supposing an extra property introduced in **[2]** that we will explain now. Every orientation preserving homeomorphism $\phi$ of $\mathbb{R}^2$ is isotopic to the identity. Fix an isotopy $I = (\phi_t)_{t\in[0,1]}$ from $\phi_0 = \mathrm{Id}$ to $\phi_1 = \phi$ and note $I_z : t \mapsto \phi_t(z)$ the trajectory of a point $z \in \mathbb{R}^2$ along the isotopy. One can define the linking number $\mathrm{Link}_I(z, z') \in \mathbb{R}$ of two different points of $\phi$ by setting

$$\mathrm{Link}_I(z, z') = \int_{I_z - I_{z'}} d\theta,$$

where $d\theta = \frac{1}{2\pi}\frac{xdy-ydx}{x^2+y^2}$ is the usual angular form ($x$ and $y$ being the cartesian coordinates on $\mathbb{R}^2$) and $I_z - I_{z'} : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ is the path defined by $(I_z - I_{z'})(t) = \phi_t(z) - \phi_t(z')$. If $I'$ is another isotopy from Id to $\phi$, there exists an integer $k \in \mathbb{Z}$ such that $\mathrm{Link}_{I'}(z, z') = \mathrm{Link}_I(z, z') + k$ for every pair of distinct points. Note that $\mathrm{Link}_I(z, z') \in \mathbb{Z}$ if $z$ and $z'$ are fixed points of $\phi$ because $I_z - I_{z'}$ is a closed path in that case. We denote $\Delta = \{(z, z') \in \mathbb{R}^2 \times \mathbb{R}^2 \mid z = z'\}$ and consider the following condition, that depends only on $\phi$:

**(P1)** the map $\mathrm{Link}_I$ is uniformly bounded on $(\mathrm{Fix}(\phi) \times \mathrm{Fix}(\phi)) \setminus \Delta$.

The notation $\mathrm{Fix}(\phi)$ stands for the fixed point set of $\phi$. The second theorem is the following:

***Theorem 2.*** — *Let $G$ be a nilpotent subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that preserves a Borel probability measure $\mu$. Suppose that $\phi$ satisfies* **(P1)** *for any $\phi \in G$. Then $G$ has a global fixed point.*

This theorem was proved by Béguin, Le Calvez, Firmo and Miernowski **[2]** under three additional hypotheses, namely:
- $G$ is abelian,
- $G$ is finitely generated,
- every element of $G$ satisfies the property **(P2)**.

This last property can be stated as follows

**(P2)** the function $\mathrm{Turn}_I : z \mapsto \int_{I_z} d\theta$ is constant in $W \cap \mathrm{Fix}(\phi)$ for some neighborhood $W$ of $\infty$.

Their proof shows the existence of a bounded $G$-orbit if the support of the measure is not contained in the set $\mathrm{Fix}(G) := \bigcap_{\phi\in G} \mathrm{Fix}(\phi)$ of global fixed points of $G$. It remains to apply the version of Theorem 1 proved by Franks, Handel and Parwani in the case of a finitely generated abelian subgroup. Of course, to get Theorem 2 we will benefit from Theorem 1 but we will have to replace **(P2)** with a weaker condition **(P2)**$'$ that we use to show that there exists a bounded $G$-orbit if the support of $\mu$ is not contained in $\mathrm{Fix}(G)$.

An interesting situation where properties **(P1)** and **(P2)** are satified is the case where $\phi$ extends to a diffeomorphism $\overline{\phi}$ of the 2-sphere. Set $\mathbb{S}^2 = \mathbb{R}^2 \sqcup \{\infty\}$ and write $\mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$ for the subgroup of $\mathrm{Diff}^1_+(\mathbb{R}^2)$ that consists of diffeomorphisms whose natural extension to $\mathbb{S}^2$ is a diffeomorphism of class $C^1$. More precisely, $\phi$ belongs to $\mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$ if the map $z \mapsto 1/\phi(1/z)$, defined if $z$ is close to 0, extends to a diffeomorphism in a neighborhood of 0. One can blow-up $\infty$ adding to $\mathbb{R}^2$ the circle $S_\infty$ of half lines of the tangent plane $T_\infty\mathbb{S}^2$ in such a way that every $\phi \in \mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$ extends to the compact space $\mathbb{R}^2 \sqcup \mathbb{S}_\infty$ in a homeomorphism that coincides with the natural action of $D\overline{\phi}(\infty)$ on $S_\infty$. If the extension has no fixed point on $S_\infty$, the fixed point set of $\phi$ is a compact subset of $\mathbb{R}^2$ and **(P1)** and **(P2)** are obviously satisfied. If it has a fixed point, every isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to $\phi$ is homotopic (relative to the ends) to an isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$. One gets a natural isotopy on $S_\infty$ defined by the action of $D\overline{\phi}(\infty)$. Say that $I$ is adapted if the time one map of the lift of this isotopy to the universal covering of $S_\infty$ has a fixed point (or equivalently if the real rotation number of the time one map is equal to 0). Say that an isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to $\phi$ is adapted if it is homotopic (relative to the ends) to such an isotopy in $\mathscr{D}$, where $\mathscr{D}$ is the set of diffeomorphisms $\phi \in \mathrm{Diff}^1_+(\mathbb{S}^2,\infty)$ such that $D\overline{\phi}(\infty)$ has real positive eigenvalues. The function $\mathrm{Link}_I$ does not depend on the choice of the adapted isotopy $I$ and will be written $\mathrm{Link}_\phi$. If $z$ is a fixed point of $\phi \in \mathrm{Diff}^1_+(\mathbb{R}^2)$, every isotopy in $\mathrm{Diff}^1_+(\mathbb{R}^2)$ from Id to $\phi$ is homotopic (relative to the ends) to an isotopy $I = (\phi_t)_{t\in[0,1]}$ in $\mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$ that fixes $z$. Similarly, one can blow up $z$ adding to $\mathbb{R}^2$ the circle $S_z$ of half lines of the tangent plane $T_z\mathbb{R}^2$ and look at the natural action of $D\overline{\phi}_t(z)$ on $S_z$. One denotes $\tau_I(z)$ the real rotation number of the time one map. It depends only on the homotopy class of $I$ in $\mathrm{Diff}^1_+(\mathbb{R}^2)$. If $\phi$ belongs to $\mathscr{D}$ and $\overline{\phi}$ has a fixed point on $S_\infty$ we will write $\tau_\phi(z) = \tau_I(z)$ if $I$ is adapted. The following result gives more information than Theorem 2 in case we are working in $\mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$. If not stated exactly as below, it was already shown in **[2]** in the case of a finitely generated abelian group. By definition a fixed point free group is a group with no global fixed points.

***Theorem 3***. — *Let $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2,\infty)$ be a fixed point free nilpotent group and $\overline{G}$ the group of extensions to $\mathbb{R}^2 \sqcup \mathbb{S}_\infty$. Then:*

1. *$\overline{G}$ has a fixed point on $S_\infty$;*
2. *if $\phi \in G$, then $\mathrm{Fix}(\phi)$ is either empty or unbounded;*
3. *every finite invariant measure of $\phi \in G$ is supported on $\mathrm{Fix}(\phi)$ and consequently every periodic point of $\phi$ is fixed;*
4. *for every $\phi \in G$ and every $z \in \mathrm{Fix}(\phi)$, one has $\tau_\phi(z) = 0$;*
5. *every $\phi \in G$ is isotopic to the identity relative to $\mathrm{Fix}(\phi)$;*

6. *for every* $\phi \in G$, *the function* $\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Link}_\phi \circ (\phi^k \times \phi^k)$ *converges uniformly to* $0$ *on* $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$.

Let us explain now the plan of the article. We will prove Theorem 3 in Section 2 assuming Theorem 1. Among the three theorems, this is the less technical one. Up to a slight modification of the developed mathematical objects, the arguments used in **[2]** for the abelian case have a natural extension to the nilpotent case.

Theorem 1 will be proved in Sections 3 and 4. As said before, it was proved for finitely generated groups **[10]**, as the following simple corollary using the finite intersection property for compact sets, where the hypothesis on the finite generation of $G$ is replaced by a compactness property. This result will be used very often in the article and we recall the proof.

***Theorem 4*** **([10])**. — *Let* $G$ *be a nilpotent subgroup of* $\mathrm{Diff}^1_+(\mathbb{R}^2)$. *Suppose there exists* $\phi \in G$ *such that* $\mathrm{Fix}(\phi)$ *is a non-empty compact set. Then* $G$ *has a global fixed point.*

*Proof.* — Let us begin by proving the theorem in case $G$ is finitely generated. Write

$$Z^{(0)}(G) = \{\mathrm{Id}\} \lhd Z^{(1)}(G) \lhd \cdots \lhd Z^{(r-1)}(G) \lhd Z^{(r)}(G) = G$$

for the ascending central series, defined inductively by the property that $Z^{(k+1)}(G)/Z^{(k)}(G)$ is the center of $G/Z^{(k)}(G)$, the integer $r$ being the nilpotency class of $G$. Consider the following subnormal series

$$\langle \phi \rangle = \langle \phi, Z^{(0)}(G)\rangle \lhd \langle \phi, Z^{(1)}(G)\rangle \lhd \cdots \lhd \langle \phi, Z^{(r-1)}(G)\rangle \lhd \langle \phi, Z^{(r)}(G)\rangle = G.$$

To get the result, let us prove inductively on $s \in \{0, \ldots, r\}$ that $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle \neq \emptyset$. By assumption the property is true for $s = 0$. Let us suppose that it is true for $s < r$. The set $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle$ is invariant by $\langle \phi, Z^{(s+1)}(G)\rangle$ because $\langle \phi, Z^{(s)}(G)\rangle$ is a normal subgroup of $\langle \phi, Z^{(s+1)}(G)\rangle$. Moreover $\mathrm{Fix}\langle \phi, Z^{(s)}(G)\rangle$ is non-empty by hypothesis, and compact because is included in $\mathrm{Fix}(\phi)$. Since every subgroup of a finitely generated nilpotent group is finitely generated (cf. **[11**, 5.2.17**]**), it follows that the group $\langle \phi, Z^{(s+1)}(G)\rangle$ is finitely generated. Since $\langle \phi, Z^{(s+1)}(G)\rangle$ is also nilpotent and preserves a non-empty compact set, it has a global fixed point by Theorem 1 for the finitely generated case **[10]**.

In the general case the first part of the proof implies that $\mathrm{Fix}(\Gamma)$ is a non-empty compact set for any finitely generated subgroup $\Gamma$ of $G$ containing $\phi$. Given finitely generated subgroups $\Gamma_1, \ldots, \Gamma_m$ of $G$ containing $\phi$, the intersection $\bigcap_{i=1}^n \mathrm{Fix}(\Gamma_i)$ is the fixed point set of the finitely generated group $\langle \bigcup_{i=1}^n \Gamma_i \rangle$ and hence non-empty. Since $\mathrm{Fix}(\phi)$ is compact and the family $\{\mathrm{Fix}\langle \phi, f\rangle\}_{f\in G}$ has the finite intersection property, it follows that $\mathrm{Fix}(G) = \bigcap_{f\in G} \mathrm{Fix}\langle \phi, f\rangle$ is a non-empty compact subset of $\mathrm{Fix}(\phi)$. □

We will see later that we can replace the hypothesis on $\mathrm{Fix}(\phi)$ in Theorem 4 by a weaker compactness property. More precisely, in order to prove Theorem 1 it suffices to show the following theorem: