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HEDGEHOGS FOR NEUTRAL DISSIPATIVE GERMS OF HOLOMORPHIC DIFFEOMORPHISMS OF $(\mathbb{C}^2, 0)$

by

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Abstract. — We prove the existence of hedgehogs for germs of complex analytic diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-neutral fixed point at the origin, using topological techniques. This approach also provides an alternative proof of a theorem of Pérez-Marco on the existence of hedgehogs for germs of univalent holomorphic maps of $(\mathbb{C}, 0)$ with a neutral fixed point.

Résumé **(Hérissons pour les germes dissipatifs neutres des difféomorphismes holomorphes de** $(\mathbb{C}^2, 0)$**)**

Nous montrons l'existence de hérissons pour les germes de difféomorphismes holomorphes de $(\mathbb{C}^2, 0)$ ayant un point fixe semi-neutre à l'origine, en utilisant uniquement des techniques topologiques. Cette approche donne également une preuve alternative d'un théorème de Pérez-Marco sur l'existence de hérissons pour les germes de difféomorphismes holomorphes de $(\mathbb{C}, 0)$ ayant un point fixe neutre.

1. Introduction

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be the convergents of α given by the continued fraction algorithm. We say that α satisfies the Brjuno condition if

$$\sum_{n \geq 0} \frac{\log q_{n+1}}{q_n} < \infty. \tag{1}$$

Brjuno [1] and Rüssmann [20] showed that if α satisfies Bjruno's condition, then any holomorphic germ with a fixed point with indifferent multiplier $\lambda = e^{2\pi i \alpha}$ is linearizable. The linearization is the irrational rotation with rotation number α. Yoccoz [27] proved that Brjuno's condition is the optimal arithmetic condition that guarantees linearizability. If α does not verify inequality (1), then there exists a holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ which is non-linearizable around the origin, that is f is not

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conjugate to the linear map $z \mapsto \lambda z$ via a holomorphic change of coordinates. The origin is called a *Cremer* fixed point.

The local dynamics of a non-linearizable map with a Cremer fixed point is complex and hard to visualize. In the '90s, Pérez-Marco [**13**] proved the existence of interesting invariant compact sets near the Cremer fixed point, called hedgehogs. Using deep results from the theory of analytic circle diffeomorphisms developed by Yoccoz [**26**], Pérez-Marco [**12, 14, 11, 15, 16**] showed that even if the map on a neighborhood of the origin is not conjugate to an irrational rotation, the points of the hedgehog are recurrent and still move under the influence of the rotation. Cheraghi [**4**] built some models for the local dynamics near Cremer points for specific cases of quadratic polynomials with high type rotation numbers using the near-parabolic renormalization of Inou and Shishikura [**8**].

In this paper we show the existence of non-trivial compact invariant sets for germs of diffeomorphisms of $(\mathbb{C}^2, 0)$ with semi-indifferent fixed points. The proof is purely topological and also provides an alternative proof for the existence of hedgehogs in dimension one.

A fixed point x of a holomorphic germ f of $(\mathbb{C}^2, 0)$ is *semi-indifferent* (or *semi-neutral*) if the eigenvalues λ and μ of the linear part of f at x satisfy $|\lambda| = 1$ and $|\mu| < 1$. In analogy with the one-dimensional dynamics, a semi-indifferent fixed point can be semi-parabolic, semi-Siegel or semi-Cremer, which essentially depends on the arithmetic properties of the neutral eigenvalue λ. We say that an isolated fixed point x is *semi-parabolic* if $\lambda = e^{2\pi i \alpha}$ and the angle $\alpha = p/q$ is rational. If α is irrational and there exists an injective holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2$ such that $f(\varphi(\xi)) = \varphi(\lambda \xi)$, for $\xi \in \mathbb{D}$, we call the fixed point *semi-Siegel*. Finally, if α is irrational and there does not exist an invariant disk on which the map is analytically conjugate to an irrational rotation, then the fixed point is called *semi-Cremer*. Note that in the latter case α does not satisfy the Brjuno condition (1).

Let E^s and E^c denote the eigenspaces of df_0 corresponding to the dissipative eigenvalue μ and respectively to the neutral eigenvalue λ. Let B' be a neighborhood of 0 and let E^s_x and E^c_x be not necessarily invariant continuous distributions such that $E^s_0 = E^s$, $E^c_0 = E^c$, and $T_x B' = E^s_x \oplus E^c_x$ for all $x \in B'$. We define the vertical cone $\mathcal{C}^v_x$ to be the set of vectors in the tangent space at x that make an angle less than or equal to α with E^s_x, for some $\alpha > 0$. The horizontal cone $\mathcal{C}^h_x$ is defined in the same way, with respect to E^c_x.

The map f is *partially hyperbolic* [(1)] on B' (see Pesin [**17**]) if there exist two real numbers $\overline{\mu}$ and $\underline{\lambda}$ such that $0 < |\mu| < \overline{\mu} < \underline{\lambda} < 1$ and a family of invariant cone fields $\mathcal{C}^{h/v}$ on B',

$$(2) \qquad df_x(\mathcal{C}^h_x) \subset \operatorname{Int} \mathcal{C}^h_{f(x)} \cup \{0\}, \; df^{-1}_{f(x)}(\mathcal{C}^v_{f(x)}) \subset \operatorname{Int} \mathcal{C}^v_x \cup \{0\},$$

(1) In this defintion, adapted to our purposes, there is no unstable bundle.

such that for every $x \in B'$ we have

$$\underline{\lambda}\,\|v\| \le \|df_x(v)\| \le 1/\underline{\lambda}\,\|v\|, \text{ for } v \in \mathcal{C}_x^h, \tag{3}$$

$$\|df_x(v)\| \le \overline{\mu}\,\|v\|, \text{ for } v \in \mathcal{C}_x^v, \tag{4}$$

for some Riemannian metric $\|\cdot\|$.

If f is partially hyperbolic, then the rate of contraction along E_x^s dominates the behavior of df_x along the complementary direction E_x^c. This domination ensures the existence of local center manifolds $W^c_{\mathrm{loc}}(0)$ relative to a neighborhood B' of 0 as graphs of functions $\varphi_f : E^c \cap B' \to E^s$, as discussed in Section 2.

Theorem A. — *Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}^2, 0)$ with a semi-indifferent fixed point at 0 with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. There exists a neighborhood $B' \subset \mathbb{C}^2$ of 0 on which f is partially hyperbolic such that for any open ball $B \Subset B'$ centered at 0 there exists a set $\mathcal{H} \subset \overline{B}$ with the following properties:*

a) *$\mathcal{H} \Subset W^c_{\mathrm{loc}}(0)$, where $W^c_{\mathrm{loc}}(0)$ is any local center manifold of the fixed point 0 constructed relative to B'.*

b) *$\mathcal{H}$ is compact, connected, completely invariant and full.*

c) *$0 \in \mathcal{H}$, $\mathcal{H} \cap \partial B \neq \emptyset$.*

d) *Every point $x \in \mathcal{H}$ has a well defined local strong stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(x)$, consisting of points from B whose orbits converge exponentially fast to the orbit of x. The strong stable set of $\mathcal{H}$ is laminated by vertical-like holomorphic disks.*

We say that $\mathcal{H}$ is *completely invariant* if $f(\mathcal{H}) \subset \mathcal{H}$ and $f^{-1}(\mathcal{H}) \subset \mathcal{H}$. The set $\mathcal{H}$ is *full* if its complement in $W^c_{\mathrm{loc}}(0)$ is connected. The local strong stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(x)$ of a point $x \in \mathcal{H}$ is defined as the set

$$\{y \in B : f^n(y) \in B \ \forall n \ge 1, \ \lim_{n\to\infty} \mathrm{dist}(f^n(y), f^n(x))/\overline{\mu}^n = 0\},$$

where $\overline{\mu}$ is the constant of partial hyperbolicity from (4).

We call the set $\mathcal{H}$ from Theorem A a *hedgehog*. The most intriguing case is when the argument of λ is irrational and $\mathcal{H}$ is not contained in the closure of a linearization domain. This happens for instance, when the origin is semi-Cremer. Theorem A is applicable to the local study of dissipative polynomial automorphisms of $\mathbb{C}^2$ with a semi-indifferent fixed point.

The theorem generalizes directly to the case of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$, for $n > 2$, which have a fixed point at the origin with exactly one eigenvalue on the unit circle and $n - 1$ eigenvalues inside the unit disk.

One interesting fact is that in this article we give a topological proof for the existence of hedgehogs in all dimensions, without using tools such as the Uniformization Theorem which are applicable only in the one-dimensional case. This approach encourages to look for topological methods that can be applied to germs with more than one neutral eigenvalue. In a second article **[10]** we use tools from complex differential geometry and quasiconformal theory (e.g., the Measurable Riemann Mapping Theorem) to study fine properties of the hedgehog. In principle, we could do everything by the latter methods, but it is instructive to know when such powerful tools are actually needed and when they can be bypassed by softer arguments.

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2. Center manifolds of the semi-indifferent fixed point

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (\lambda x + f_1(x, y), \mu y + f_2(x, y))$ be a holomorphic germ with a semi-indifferent fixed point at the origin. We also refer to f as a neutral dissipative germ of $(\mathbb{C}^2, 0)$.

The semi-indifferent fixed point has a well-defined unique analytic strong stable manifold $W^{\mathrm{ss}}(0)$ corresponding to the dissipative eigenvalue μ. It consists of points that are attracted to 0 exponentially fast, and defined as

$$W^{\mathrm{ss}}(0) := \{x \in \mathbb{C}^2 : \lim_{n\to\infty} \operatorname{dist}(f^n(x), 0)/\mu^n = \text{const.}\}. \tag{5}$$

The semi-indifferent fixed point also has a (non-unique) center manifold $W^c_{\mathrm{loc}}(0)$ of class C^k for some integer $k \geq 1$, tangent at 0 to the eigenspace E^c of the neutral eigenvalue λ. There exists a ball B_δ (where the size of δ depends on k) centered at the origin in which the center manifold is locally the graph of a C^k function $\varphi_f : E^c \to E^s$ and has the following properties:

a) *Local Invariance:* $f(W^c_{\mathrm{loc}}(0)) \cap B_\delta \subset W^c_{\mathrm{loc}}(0)$.
b) *Weak Uniqueness:* If $f^{-n}(x) \in B_\delta$ for all $n \in \mathbb{N}$, then $x \in W^c_{\mathrm{loc}}(0)$. Thus center manifolds may differ only on trajectories that leave the neighborhood B_δ under backward iterations.
c) *Shadowing:* Given any point x such that $f^n(x) \to 0$ as $n \to \infty$, there exists a positive constant k and a point $y \in W^c_{\mathrm{loc}}(0)$ such that $\|f^n(x) - f^n(y)\| < k\overline{\mu}^n$ as $n \to \infty$. In other words, every orbit which converges to the origin can be described as an exponentially small perturbation of some orbit on the center manifold.

Consider the space of holomorphic germs g of $(B_\delta, 0)$ which are C^k-close to f such that g has a semi-indifferent fixed point at the origin. We will later consider a sequence of germs with a semi-parabolic fixed point which converges uniformly to a germ with a semi-Cremer fixed point. Proposition 2.1 shows that even if the center manifold of g is not unique, it may be chosen to depend continuously on g for the C^k topology. Let $E^c(\delta) = E^c \cap B_\delta$.

Proposition 2.1. — *The map g has a C^k center manifold defined as the graph of a C^k function $\varphi_g : E^c(\delta) \to E^s(\delta)$ such that the map $(g, x) \mapsto \varphi_g(x)$ is C^k with respect to g.*