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STABLE ACCESSIBILITY WITH 2-DIMENSIONAL CENTER

by

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Abstract. — For partially hyperbolic diffeomorphisms with 2-dimensional center, accessibility is C^1-stable. Moreover, for center bunched skew-products (stable) accessibility is C^∞-dense.

Résumé **(Accessibilité stable de dimension centrale 2).** — L'accessibilité est une propriété C^1-stable parmi les difféomorphismes partiellement hyperboliques à dimension centrale 2. De plus, l'accessibilité (stable) est une propriété C^1-dense dans le domaine des produits gauches satisfaisant la condition de regroupement central ('center bunching').

1. Introduction

A diffeomorphism $f : M \to M$ of a compact manifold M is *partially hyperbolic* if there exist: a continuous splitting of the tangent bundle $TM = E^u \oplus E^c \oplus E^s$ invariant under the derivative Df (all three sub-bundles are assumed to have positive dimension); a Riemannian metric $\|\cdot\|$ on M; and positive continuous functions ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with ν, $\hat{\nu} < 1$ and $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$, such that

$$\begin{aligned} \|Df(p)v\| &< \nu(p) && \text{if } v \in E^s(p), \\ \gamma(p) < \|Df(p)v\| &< \hat{\gamma}(p)^{-1} && \text{if } v \in E^c(p), \\ \hat{\nu}(p)^{-1} &< \|Df(p)v\| && \text{if } v \in E^u(p) \end{aligned} \tag{1}$$

for any unit vector $v \in T_pM$. This is an open property in the space of C^1 diffeomorphisms. We will denote $d_* = \dim E^*$, for $* \in \{u, c, s\}$, and $d = \dim M$.

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The stable bundle E^s and the unstable bundle E^u are uniquely integrable and their integral manifolds form two quasi transverse continuous foliations, $\mathcal{W}^u = \mathcal{W}^u_f$ and $\mathcal{W}^s = \mathcal{W}^s_f$, whose leaves are immersed submanifolds of the same class of differentiability as f. These are called the *strong unstable* and *strong stable* foliations of f. They are invariant under f, in the sense that $f(\mathcal{W}^*(x)) = \mathcal{W}^*(f(x))$ for any $x \in M$ and $* \in \{u, s\}$. Given $\epsilon > 0$ and $* \in \{u, s\}$, we represent by $\mathcal{W}^*_\epsilon(x) = \mathcal{W}^*_{f,\epsilon}(x)$ the ϵ-neighborhood of x inside $\mathcal{W}^*(x)$.

Given two points $x, y \in M$, we say that x is accessible from y if there exists a C^1 path that connects x to y and is tangent at every point to the union $E^u \cup E^s$. The equivalence classes of this (equivalence) relation are called *f-accessibility classes.* The diffeomorphism f is called *accessible* if there exists a unique f-accessibility class, namely, the ambient M. Moreover, f is called *stably accessible* if it admits a C^1 neighborhood $\mathcal{U}$ such that every C^2 diffeomorphism $g \in \mathcal{U}$ is accessible.

For any $k \geq 1$, we denote by $\mathcal{PH}^k$ the space of C^k partially hyperbolic diffeomorphisms in M. Most of our results concern the subspace $\mathcal{PH}^k_2$ of diffeomorphisms $f \in \mathcal{PH}^k$ with 2-dimensional center bundle, that is, such that $d_c = 2$.

Theorem A. — *If $f \in \mathcal{PH}^1_2$ is accessible then f is stably accessible.*

We say that an f-accessibility class C is *stable* if for every compact set $K \subset C$ there exists a C^1 neighborhood $\mathcal{U} = \mathcal{U}_K$ of f such that K is contained in a unique g-accessibility class for every C^2 diffeomorphism $g \in \mathcal{U}$. In particular, f is stably accessible if, and only if, the ambient M is a stable f-accessibility class.

Stable accessibility classes are open sets. Indeed, let p and q be two distinct points in C (for instance, in the same stable manifold). For any $r \in M$ close to q, let $h : M \to M$ be a diffeomorphism C^∞ close to the identity, such that $h(p) = p$ and $h(r) = q$. Then $g = h \circ f \circ h^{-1}$ is close to f. Taking $K = \{p, q\}$, the assumption implies that p and q are in the same g-accessibility class. This means that p and q are in the same f-accessibility class, that is, $r \in C$. So, C contains a whole neighborhood of q.

Here we prove that the converse is also true, at least when the center bundle is 2-dimensional:

Theorem B. — *If $f \in \mathcal{PH}^1_2$ then any open f-accessibility class is stable.*

Theorem A is a direct consequence of Theorem B. The main technical step in the proof of Theorem B is a result on approximation of general paths in accessibility classes by a certain class of paths for which a continuation exists for every nearby diffeomorphism. This result is stated in Section 4 (Theorem 4.1), where we also explain how it leads to Theorem B.

In Sections 6–5 we state and prove a result about density of stable accessibility (Theorem 6.1), for a class of fibered partially hyperbolic diffeomorphisms with 2-dimensional center bundle. It contains a claim made in Section 7 of our paper [**2**], that was used for proving Theorem H in that paper. After our research had been completed, we learned from V. Horita and M. Sambarino that they had independently obtained a similar result, in a paper that appeared in [**8**].

When the center dimension $d_c = 1$, the accessibility property is always stable [**5**]. The present work extends that fact to center dimension equal to 2. Recently, and also in the 2-dimensional case, J. Rodriguez-Hertz and C. Vasquez [**13**] proved that accessibility classes are immersed submanifolds, which implies Theorem A.

When the center bundle is one-dimensional, the (stable) accessibility property is known to be C^r dense among partially hyperbolic diffeomorphisms [**3, 12**]. Without any hypothesis on the dimension of the central bundle, Dolgopyat and Wilkinson [**6**] proved that stable accessibility is C^1 dense.

2. Deformations paths

In this section, all maps are assumed to be C^1 and proximity is always meant in the C^1 topology. We introduce a class of paths, that we call deformation paths, contained in accessibility classes and having a useful property of persistence under variation of the diffeomorphism and the base point. This also provides a kind of parametrization for accessibility classes:

Theorem 2.1. — *For every $f \in \mathscr{PH}^1$, there exist $k \geq 1$, a neighborhood $\mathscr{V}$ of f and a sequence $P_l : \mathscr{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \to M$ of continuous maps such that, for any $g \in \mathscr{V}$,*

1. *$P_m(g,\cdot,w) \circ P_l(g,\cdot,v) = P_{l+m}(g,\cdot,(v,w))$ for every $l \geq 1$ and $m \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$ and $w \in \mathbb{R}^{k(d_u+d_s)m}$;*
2. *$\zeta \mapsto P_l(g,\zeta,v)$ is a homeomorphism from M to M, with $P_l(g,\cdot,0) = \mathrm{id}$, for every $l \geq 1$ and $v \in \mathbb{R}^{k(d_u+d_s)l}$;*
3. *$\bigcup_{n\geq 1} P_n(\{(g,z)\} \times \mathbb{R}^{k(d_u+d_s)n})$ is the g-accessibility class of each $z \in M$.*

A *deformation path based on* (f,z) is a (continuous) map $\gamma : [0,1] \to M$ such that there exist $l \geq 1$ and a continuous map $\Gamma : [0,1] \to \mathbb{R}^{k(d_u+d_s)l}$ satisfying $\gamma(t) = P_l(f,z,\Gamma(t))$. Notice that any deformation path based on (f,z) is contained in the f-accessibility class of z. It follows immediately from the definition that deformation paths are persistent, in the following sense:

Corollary 2.2. — *If $\gamma : [0,1] \to M$ is a deformation path based on (f,z) then, for any g close to f and any w close to z, there exists a deformation path based on (g,w) that is uniformly close to γ.*

In the remainder of this section we prove Theorem 2.1. Let $I = [-1,1]$. We need the following particular case of [**7**, Theorem 4.1]:

Lemma 2.3. — *For every $f \in \mathscr{PH}^1$ and $\zeta \in M$, there exists a neighborhood $\mathscr{V}$ of f and a continuous map $\psi = \psi_{f,\zeta} : \mathscr{V} \times I^d \to M$ such that for every $g \in \mathscr{V}$,*

1. *$\psi(g,0) = \zeta$,*

2. *$x \mapsto \psi(g,x)$ is a homeomorphism,*
3. *$\psi(g,x,y) \in W^u_g(\psi(g,0,y))$ for every $x \in I^{d_u}$ and $y \in I^{d-d_u}$.*

Lemma 2.4. — *For every $f \in \mathcal{PH}^1$ there exist a neighborhood $\mathcal{V}$ of f, numbers $k \geq 1$ and $\epsilon > 0$ and continuous maps $\Phi_u : \mathcal{V} \times M \times \mathbb{R}^{kd_u} \to M$ and $\Phi_s : \mathcal{V} \times M \times \mathbb{R}^{kd_s} \to M$ such that:*

1. *$x \mapsto \Phi_u(g,x,v)$ is a homeomorphism, for every $g \in \mathcal{V}$ and $v \in \mathbb{R}^{kd_u}$;*
2. *$\mathcal{W}^u_{g,\epsilon}(x) \subset \Phi_u(\{g\} \times \{x\} \times \mathbb{R}^{kd_u}) \subset W^u_g(x)$ for every $g \in \mathcal{V}$ and $x \in M$,*

and analogously for Φ_s.

Proof. — We will only go through the details of the construction of Φ_u, the case of Φ_s being analogous. Let $h_t : I \to I$ be the flow satisfying $(dh_t/dt)(x) = 1 - h_t(x)^2$. Let $H : I^d \to I$ be given by $H(v) = (1 - v^2_{d_u+1}) \dots (1 - v^2_d)$. For $v \in \mathbb{R}^{d_u}$, let $h_v : I^d \to I^d$ be given by

$$h_v(x) = \big(h_{H(x)v_1}(x_1), \dots, h_{H(x)v_{d_u}}(x_{d_u}), x_{d_u+1}, \dots, x_d\big).$$

Pick points $\zeta_i \in M$, $1 \leq i \leq k$ so that the interiors of the images $\psi_i(\{f\} \times I^d)$ cover M, where $\psi_i = \psi_{f,\zeta_i} : \mathcal{V}_i \times I^d \to M$ are the maps given by Lemma 2.3. Let $\mathcal{V}$ be a neighborhood of f contained in $\bigcap_i \mathcal{V}_i$ and ϵ be a positive number such that for every $g \in \mathcal{V}$ and $z \in M$ there exist i and y such that

$$W^u_{g,\epsilon}(z) \subset \psi_i\big(\{g\} \times \mathrm{inter}(I^{d_u}) \times \{y\}\big).$$

Let $\Phi_i : \mathcal{V} \times M \times \mathbb{R}^{d_u} \to M$ be given by

$$\Phi_i(g, \psi_i(g,\zeta), v) = \psi_i(g, h_v(\zeta)) \quad \text{for } \zeta \in I^d$$
$$\Phi_i(g,z,v) = z \quad \text{if } z \notin \psi_i(\{g\} \times I^d).$$

Then define $\Phi^{(i)} : \mathcal{V} \times M \times \mathbb{R}^{id_u} \to M$, $1 \leq i \leq k$ by

$$\Phi^{(1)} = \Phi_1 \quad \text{and} \quad \Phi^{(i+1)}(g, \cdot, (w_i, w)) = \Phi_{i+1}(g, \cdot, w) \circ \Phi^{(i)}(g, \cdot, w_i)$$

and take $\Phi_u = \Phi^{(k)}$. Claim (1) follows from part (2) of Lemma 2.3, by composition. The lower bound in claim (2) follows from the choice of ϵ and the upper bound is a consequence of part (3) of Lemma 2.3. □

Proof of Theorem 2.1. — Define $P_l : \mathcal{V} \times M \times \mathbb{R}^{k(d_u+d_s)l} \to M$, $l \in \mathbb{N}$ by letting

$$P_1(g, \cdot, (w_u, w_s)) = \Phi_s(g, \cdot, w_s) \circ \Phi_u(g, \cdot, w_u)$$

for $w_u \in \mathbb{R}^{kd_u}$ and $w_s \in \mathbb{R}^{kd_s}$ and

$$P_l(g, \cdot, (w_1, \dots, w_l)) = P_1(g, \cdot, w_l) \circ \dots \circ P_1(g, \cdot, , w_1)$$

for $w_1, \dots, w_l \in \mathbb{R}^{k(d_u+d_s)}$. Property (1) in Theorem 2.1 is a direct consequence of this definition. Property (2) follows from part (2) of Lemma 2.3, by composition. Finally, Lemma 2.4 gives that $\bigcup_{n \geq 1} P_n(g, z, \mathbb{R}^{k(d_u+d_s)n})$ is the g-accessibility class of z, as claimed in part (3) of the theorem. □