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THE VERIFICATION OF THE NIRENBERG-TREVES CONJECTURE

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Abstract. — *In a series of recent papers, Nils Dencker proves that condition* (ψ) *implies the local solvability of principal type pseudodifferential operators (with loss of* $\frac{3}{2}+\epsilon$ *derivatives for all positive* ϵ*), verifying the last part of the Nirenberg-Treves conjecture, formulated in 1971. The origin of this question goes back to the Hans Lewy counterexample, published in 1957. In this text, we follow the pattern of Dencker's papers, and we provide a proof of local solvability with a loss of* $\frac{3}{2}$ *derivatives.*

INTRODUCTION

The Hans Lewy counterexample. — In 1957, Hans Lewy stunned the mathematical world by showing that very simple and natural linear PDE could fail to have solutions. The Hans Lewy operator L_0, introduced in [30], is the following complex vector field in $\mathbb{R}^3$

$$L_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}. \tag{0.1}$$

There exists $f \in C^\infty$ such that the equation $L_0 u = f$ has no distribution solution, even locally. This discovery came as a great shock for several reasons. First of all, L_0 has a very simple expression and is natural as the Cauchy-Riemann operator on the boundary of the pseudo-convex domain

$$\{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + 2\Im z_2 < 0\}.$$

Moreover L_0 is a non-vanishing vector field so that no pathological behaviour related to multiple characteristics is to be expected. In the fifties, it was certainly the conventional wisdom that any "reasonable" operator should be locally solvable, and obviously (0.1) was indeed very reasonable, so the conclusion was that the CW should be revisited(1). One of the questions posed by such a counterexample was to find some geometric explanation for this phenomenon. This was done in 1960 by L. Hörmander

(1)Gaston Bachelard did not use the words conventional wisdom, but wrote in [1] "La science, dans son besoin d'achèvement comme dans son principe, s'oppose absolument à l'opinion. S'il lui arrive, sur un point particulier, de légitimer l'opinion, c'est pour d'autres raisons que celles qui fondent l'opinion ; de sorte que l'opinion a, en droit, toujours tort. L'opinion pense mal ; elle ne pense pas : elle traduit des besoins en connaissances. En désignant les objets par leur utilité, elle s'interdit de les connaître."

in [15] who proved that if p is the symbol of a differential operator such that, at some point (x, ξ) in the cotangent bundle,

$$p(x,\xi)=0 \quad \text{and} \quad \{\Re p, \Im p\}(x,\xi)>0, \tag{0.2}$$

then the operator P with principal symbol p is not locally solvable at x; in fact, there exists $f \in C^\infty$ such that, for any neighborhood V of x the equation $Pu = f$ has no solution $u \in \mathcal{D}'(V)$. Of course, in the case of differential operators, the sign > 0 in (0.2) can be replaced by $\neq 0$ since the Poisson bracket $\{\Re p, \Im p\}$ is then an homogeneous polynomial with odd degree in the variable ξ. Nevertheless, it appeared later (in [16]) that the same statement is true for pseudodifferential operators, so we keep it that way. Since the symbol of $-iL_0$ is $\xi_1 - x_2\xi_3 + i(\xi_2 + x_1\xi_3)$, and the Poisson bracket $\{\xi_1 - x_2\xi_3, \xi_2 + x_1\xi_3\} = 2\xi_3$, the assumption (0.2) is fulfilled for L_0 at any point x in the base and the nonsolvability property follows. This gives a necessary condition for local solvability of pseudodifferential equations: a locally solvable operator P with principal symbol p must satisfy

$$\{\Re p, \Im p\}(x,\xi) \le 0 \quad \text{at} \quad p(x,\xi)=0. \tag{0.3}$$

Naturally, condition (0.3) is far from being sufficient for solvability (see e.g. the nonsolvable M_3 below in (0.4)). After the papers [30], [15], the curiosity of the mathematical community was aroused in search of a geometric condition on the principal symbol, characterizing local solvability of principal type operators. It is important to note that for principal type operators with a real principal symbol, such as a non-vanishing real vector field, or the wave equation, local solvability was known after the 1955 paper of L. Hörmander [14]. In fact these results extend quite easily to the pseudodifferential real principal type case. As shown by the Hans Lewy counterexample and the necessary condition (0.3), the matters are quite different for complex-valued symbols.

Some examples. — It is certainly helpful to take a look now at some simple models. For $t, x \in \mathbb{R}$, with the usual notations

$$D_t = -i\partial_t, \quad (\widehat{|D_x|u})(\xi) = |\xi|\hat{u}(\xi),$$

where $\hat{u}$ is the x-Fourier transform of u, $l \in \mathbb{N}$, let us consider the operators defined by

$$M_l = D_t + it^l D_x, \quad N_l = D_t + it^l |D_x|. \tag{0.4}$$

It is indeed rather easy to prove that, for $k \in \mathbb{N}$, $M_{2k}, N_{2k}, N^*_{2k+1}$ are solvable whereas M_{2k+1}, N_{2k+1} are nonsolvable. In particular, the operators M_1, N_1 satisfy (0.2). On the other hand, the operator $N_1^* = D_t - it|D_x|$ is indeed solvable since its adjoint operator N_1 verifies the *a priori* estimate

$$T\|N_1 u\|_{L^2(\mathbb{R}^2)} \ge \|u\|_{L^2(\mathbb{R}^2)},$$

for a smooth compactly supported u vanishing for $|t| \geq T/2$. No such estimate is satisfied by $N_1^* u$ since its x-Fourier transform is

$$-i\partial_t v - it|\xi|v = (-i)(\partial_t v + t|\xi|v),$$

where v is the x-Fourier transform of u. A solution of $N_1^* u = 0$ is thus given by the inverse Fourier transform of $e^{-t^2|\xi|/2}$, ruining solvability for the operator N_1. A complete study of solvability properties of the models M_l was done in [33] by L. Nirenberg and F. Treves, who also provided a sufficient condition of solvability for vector fields; the analytic-hypoellipticity properties of these operators were also studied in a paper by S. Mizohata [31]. The simplicity of the two-dimensional models (0.4) is somewhat misleading, since they can be reduced via the Fourier transform, to the study of an ODE. It is not the case of the following examples, which are genuinely three-dimensional. The operators

$$P_{klm} = D_{x_1} - ix_1^{2k}(D_{x_2} + x_1^{2l+1} x_2^{2m}|D_x|), \quad x \in \mathbb{R}^3, k, l, m \in \mathbb{N}, \tag{0.5}$$

are locally solvable since their adjoints are subelliptic (see chapter 27 in the fourth volume of [19]). On the other hand the operators

$$D_{x_1} + ia(x)(D_{x_2} + x_1^{2l+1} x_2^{2m}|D_x|), \quad a \in C^\infty(\mathbb{R}^3; (-\infty, 0]), l, m \in \mathbb{N}, \tag{0.6}$$

are also locally solvable, but the proof is not elementary.

The expression of the Nirenberg-Treves conjecture. — Let us look first at the operator

$$L = D_t + iq(t, x, D_x), \tag{0.7}$$

where q is a real-valued first-order symbol. The symbol of L is thus $\tau + iq(t, x, \xi)$. The bicharacteristic curves of the real part are oriented straight lines with direction $\partial/\partial t$; now we examine the variations of the imaginary part $q(t, x, \xi)$ along these lines. It amounts only to check the functions $t \mapsto q(t, x, \xi)$ for fixed (x, ξ). The good cases in (0.4) (when solvability holds) are $t^{2k}\xi, -t^{2k+1}|\xi|$: when t increases these functions do not change sign from $-$ to $+$. The bad cases are $t^{2k+1}|\xi|$: when t increases these functions do change sign from $-$ to $+$; in particular, the nonsolvable case (0.2), tackled in [15], corresponds to a change of sign of $\Im p$ from $-$ to $+$ at a simple zero. The general formulation of condition (ψ) for a principal type operator with principal symbol p is as follows: for all $z \in \mathbb{C}$, $\Im(zp)$ does not change sign from $-$ to $+$ along the oriented bicharacteristic curves of $\Re(zp)$. It is a remarkable and non-trivial fact (due to the articles [3] of J.-M. Bony and [6] of H. Brézis) that this condition is invariant by multiplication by an elliptic factor. The *Nirenberg-Treves conjecture*, proved in several cases in [33], [34], [35], such as for differential operators with analytic coefficients, states that, *for a principal type pseudodifferential equation, condition* (ψ) *is equivalent to local solvability.* Using the Malgrange-Weierstrass theorem on normal forms of complex-valued non-degenerate C^∞ functions and the Egorov theorem on quantization of homogeneous canonical transformations, there is no loss of generality considering only first order operators of type (0.7). The expression of condition (ψ) for L is then very simple since it reads

$$q(t,x,\xi) < 0 \quad \text{and} \quad s > t \Longrightarrow q(s,x,\xi) \leq 0. \tag{0.8}$$

The necessity of condition (ψ) for local solvability. — In 1981, following an idea given by R.D. Moyer [32] for a result in two dimensions, L. Hörmander proved in [18] that condition (ψ) is necessary for local solvability: assuming that condition (ψ) is not satisfied for a principal type operator P, he was able to construct some approximate non-trivial solutions u for the adjoint equation $P^*u = 0$, which implies that P is not solvable. Although the construction is elementary for the model operators N_{2k+1} in (0.4) (as sketched above for N_1), the multidimensional proof is rather involved and based upon a geometrical optics method adapted to the complex case. The details can be found in the proof of theorem 26.4.7′ of [19].

The proof of the conjecture for differential operators and in 2D. — For differential operators, condition (ψ) is equivalent to ruling out any change of sign of $\Im p$ along the bicharacteristics of $\Re p$ (the latter condition is called condition (P)); this fact is due to the identity $p(x,-\xi) = (-1)^m p(x,\xi)$, valid for an homogeneous polynomial of degree m in the variable ξ. Note that the expression of condition (P) for L in (0.7) is simply $q(t,x,\xi)q(s,x,\xi) \geq 0$. In 1973, R. Beals and C. Fefferman [2] took as a starting point the aforementioned results of L. Nirenberg and F. Treves on differential operators with analytic coefficients and, removing that analyticity assumption, were able to prove the sufficiency of condition (P) for local solvability, obtaining thus the sufficiency of condition (ψ) for local solvability of differential equations. The key ingredient was a drastically new vision of the pseudodifferential calculus, devised to obtain a factorization of the function q in (0.7) of the type

$$q(t,x,\xi) = a(t,x,\xi)b(x,\xi), \quad a \leq 0 \text{ of order } 0 \text{ and } b \text{ of order } 1, \tag{0.9}$$

in regions of the phase space much smaller than cones or semi-classical "boxes" $\{(x,\xi), |x| \leq 1, |\xi| \leq h^{-1}\}$. Considering the family $\big\{q(t,x,\xi)\big\}_{t\in[-1,1]}$ of classical homogeneous symbols of order 1, they define, via a Calderón-Zygmund decomposition, a pseudodifferential calculus depending on the family $\{q(t,\cdot)\}$, in which all these symbols are first order but also such that, at some level t_0, some ellipticity property of $q(t_0,\cdot)$ or $\nabla_{x,\xi}q(t_0,\cdot)$ is satisfied. Although a factorization (0.9) can be obtained for differential operators with analytic regularity satisfying condition (ψ), such a factorization is not true in the C^∞ case, even microlocally in the standard sense(2). This is why R. Beals and C. Fefferman had to resort to a much finer microlocalization scheme

(2)Consider the C^∞ function q defined on $\mathbb{R}^3$ by $q(t,x,\xi) = \begin{cases} (\xi - te^{-1/x})^2 & \text{if } x > 0, \\ \xi(\xi - e^{1/x}) & \text{if } x < 0. \end{cases}$ For every fixed (x,ξ), the function $t \mapsto q(t,x,\xi)$ does not change sign since $q(t,x,\xi)q(s,x,\xi) \geq 0$. Nevertheless than the classical one. In fact, the proof of R. Beals and C. Fefferman marked the day when microlocal analysis stopped being only homogeneous or semi-classical, thanks to methods of harmonic analysis such as Calderón-Zygmund decomposition made compatible with the Heisenberg uncertainty principle. In 1988, N. Lerner [23] proved the sufficiency of condition (ψ) for local solvability of pseudodifferential equations in two dimensions and as well for the classical oblique-derivative problem [24]. The method of proof of these results is also based upon a factorization analogous to (0.9) but where $b(x, \xi)$ is replaced by $\beta(t, x)|\xi|$ and β is a smooth function such that $t \mapsto \beta(t, x)$ does not change sign from $+$ to $-$ when t increases. Then a properly defined sign of $\beta(t, x)$ appears as a non-decreasing operator and the Nirenberg-Treves energy method can be adapted to this situation. The Beals-Fefferman result mentioned above proved the local existence of $H^{s+m-1}_{\rm loc}$ solutions u to the equation $Lu = f$ with a source f in $H^s_{\rm loc}$, whenever L is an operator of order m satisfying condition (P); since the size of the neigbourhood where the equation is satisfied may depend on the index s, this is not enough to get C^∞ solutions whenever f is smooth. The existence of C^∞ solutions for C^∞ sources was proved by L. Hörmander in [17] for pseudodifferential equations satisfying condition (P). We refer the reader to the paper [21] for a more detailed historical overview of this problem. On the other hand, it is clear that our interest is focused on solvability in the C^∞ category. Let us nevertheless recall that the sufficiency of condition (ψ) in the analytic category (for microdifferential operators acting on microfunctions) was proved by J.-M. Trépreau [37] (see also [20], chapter VII).

Counting the loss of derivatives. — Let us consider a principal-type pseudodifferential operator L of order m. We shall say that L is locally solvable with a loss of μ derivatives whenever the equation $Lu = f$ has a local solution u in the Sobolev space $H^{s+m-\mu}_{\rm loc}$ for a source f in $H^s_{\rm loc}$. Note that the loss is zero if and only if L is elliptic. Since for the simplest principal type equation $\partial/\partial x_1$, the loss of derivatives is 1, we shall consider that 1 is the "ordinary" loss of derivatives. When L satisfies condition (P) (e.g. if L is a differential operator satisfying condition (ψ)), or when L satisfies condition (ψ) in two dimensions, the estimates

$$C\|L^* u\|_{H^s} \geq \|u\|_{H^{s+m-1}}, \tag{0.10}$$

valid for smooth compactly supported u with a small enough support, imply local solvability with loss of 1 derivative, the ordinary loss referred to above. For many years, repeated claims were made that condition (ψ) for L implies (0.10), that is solvability with loss of 1 derivative. It turned out that these claims were wrong, as shown by N. Lerner in [25] by the following result (see also section 6 in the survey article [21] by L. Hörmander). *There exists a principal type first-order pseudodifferential operator L*

one can show that it is not possible to find some C^∞ functions a, b such that a is nonnegative and b independent of t such that $q = ab$.