Séminaire BOURBAKI Mars 2006
58e année, 2005-2006, nº 961, p. 237 à 294

RIGIDITY RESULTS FOR BERNOULLI ACTIONS AND THEIR VON NEUMANN ALGEBRAS [after Sorin Popa]

by **Stefaan VAES**

Contents

1. INTRODUCTION

Suppose that a countable group G acts freely and ergodically on the standard probability space (X, μ) preserving the probability measure μ. We are interested in several types of 'isomorphisms' between such actions. Two actions are said to be

(1) *conjugate* if there exist a group isomorphism and a measure space isomorphism satisfying the obvious conjugacy formula;

(2) *orbit equivalent* if there exists a measure space isomorphism sending orbits to orbits, i.e., the *equivalence relations* given by the orbits are isomorphic;

(3) *von Neumann equivalent* if the crossed product von Neumann algebras are isomorphic.

Note that the *crossed product* construction[1] has been introduced by Murray and von Neumann [41], who called it the *group measure space construction.*

It is clear that conjugacy of two actions implies orbit equivalence. Since the crossed product von Neumann algebra can be defined directly from the equivalence relation given by the orbits, orbit equivalence implies von Neumann equivalence. *Rigidity results* provide the converse implications for certain actions of certain groups. This is a highly non-trivial matter. Dye [16, 17] proved that all free ergodic measure preserving actions of groups with polynomial growth on the standard probability space are orbit equivalent. This result was extended to all *amenable groups* by Ornstein and Weiss [45]. Finally, Connes, Feldman and Weiss [10] showed that every ergodic amenable probability measure preserving countable equivalence relation is generated by a free $\mathbb{Z}$-action and is hence unique. Summarizing, for amenable group actions all information on the group, except its amenability, gets lost in the passage to the equivalence relation.

Concerning the relation between orbit equivalence and von Neumann equivalence, it was noted by Feldman and Moore [19] that the *pair* $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes G$ remembers the equivalence relation. The abelian subalgebra $L^\infty(X,\mu)$ is a so-called *Cartan subalgebra.* So, in order to deduce orbit equivalence from von Neumann equivalence, we need certain uniqueness results for Cartan subalgebras, which is an extremely hard problem. Connes and Jones [12] gave the first examples of non orbit equivalent, yet von Neumann equivalent actions.

In this talk, we discuss Popa's recent breakthrough rigidity results for Bernoulli actions[2] of Kazhdan groups. These results open a new era in von Neumann algebra theory, with striking applications in ergodic theory. The heart of Popa's work is his *deformation/rigidity strategy*: he discovered families of von Neumann algebras with a rigid subalgebra but yet with just enough deformation properties in order for the rigid part to be uniquely determined inside the ambient algebra (up to unitary conjugacy). This leads to far reaching classification results for these families of von Neumann algebras. Popa considered the deformation/rigidity strategy for the first time in [54]. In [52], he used it to deduce orbit equivalence from mere von Neumann equivalence between certain group actions and to give the first examples of II_1 factors with trivial fundamental group, through an application of Gaboriau's ℓ^2 Betti numbers of equivalence relations [22]. Deformation/rigidity arguments are again the crucial ingredient in the papers [48, 55, 56, 53] that we discuss in this talk and they are applied in [29], in the study of amalgamated free products. These ideas may lead to

[1] The crossed product von Neumann algebra $L^\infty(X,\mu) \rtimes G$ contains a copy of $L^\infty(X,\mu)$ and a copy of the group G by unitary elements in the algebra, and the commutation relations between both are given by the action of G on (X,μ).

[2] Every discrete group G acts on $(X,\mu) = \prod_{g\in G}(X_0,\mu_0)$, by shifting the Cartesian product. Here (X_0,μ_0) is the standard non-atomic probability space and the action is called the Bernoulli action of G.

many more applications in von Neumann algebra and ergodic theory (see *e.g.* the new papers [28, 58] written since this talk was given).

In the papers discussed in this talk, the *rigidity* comes from the group side and is given by Kazhdan's property (T) [15, 36] and more generally, by the relative property (T) of Kazhdan-Margulis (see [26] and Valette's Bourbaki seminar [63] for details): the groups dealt with contain an infinite normal subgroup with the relative property (T) and are called *w-rigid groups.* Popa discovered a strong *deformation property* shared by the Bernoulli actions, and called it *malleability.* In a sense, a Bernoulli action can be continuously deformed until it becomes orthogonal to its initial position. In order to exploit the tension between the deformation of the action and the rigidity of the group, yet another technique comes in. Using *bimodules* (Connes' correspondences), Popa developed a very strong method to prove that two subalgebras of a von Neumann algebra are unitarily conjugate. Note that he used this bimodule technique in many different settings, see [29, 46, 55, 56, 52, 51].

The following are the two main results of [48, 55, 56] and are discussed below. The *orbit equivalence superrigidity theorem* states that the equivalence relation given by the orbits of a Bernoulli action of a w-rigid group, entirely remembers the group and the action. The *von Neumann strong rigidity theorem* roughly says that whenever a Bernoulli action is von Neumann equivalent with a free ergodic action of a w-rigid group, the actions are actually conjugate. It is the first theorem in the literature deducing conjugacy of actions out of von Neumann equivalence. The methods and ideas behind these far reaching results are fundamentally *operator algebraic* and yield striking theorems in *ergodic theory.*

Some important conventions

All probability spaces in this talk are standard. All actions of countable groups G on (X, μ) are supposed to preserve the probability measure μ. All statements about elements of (X, μ) only hold almost everywhere. A *w-rigid group* is a countable group that admits an infinite normal subgroup with the relative property (T).

Orbit equivalence superrigidity

In [48], the deformation/rigidity technique leads to the following orbit equivalence superrigidity theorem.

Theorem (Theorem 4.4). — *Let $G \curvearrowright (X, \mu)$ be the Bernoulli action of a w-rigid group G as above. Suppose that G does not have finite normal subgroups. If the restriction to $Y \subset X$ of the equivalence relation given by $G \curvearrowright X$ is given by the orbits of $\Gamma \curvearrowright Y$ for some group Γ acting freely and ergodically on Y, then, up to measure zero, $Y = X$ and the actions of G and Γ are conjugate through a group isomorphism.*

The theorem implies as well that the restriction to a Borel set of measure $0 < \mu(Y) < 1$, of the Bernoulli action of a w-rigid group G without finite normal subgroups, yields an ergodic probability measure preserving countable equivalence relation that cannot be generated by a free action of a group. The first examples of this phenomenon – answering a question of Feldman and Moore – were given by Furman in [21]. Dropping the ergodicity, examples were given before by Adams in [1], who also provides examples in the Borel setting.

Popa proves the orbit equivalence superrigidity for the Bernoulli action of G on X using his even stronger *cocycle superrigidity theorem*: any 1-cocycle for the action $G \curvearrowright X$ with values in a discrete group Γ is cohomologous to a homomorphism of G to Γ. The origin of orbit equivalence rigidity and cocycle rigidity theory lies in Zimmer's pioneering work. Zimmer proved in [66] his celebrated cocycle rigidity theorem and used it to obtain the first orbit equivalence rigidity results (see Section 5.2 in [67]). Since Zimmer's theorem deals with cocycles taking values in linear groups, he obtains orbit equivalence rigidity results where both groups are assumed to be linear (see [68]). Furman developed in [20, 21] a new technique and obtains an orbit equivalence superrigidity theorem with quite general ergodic actions of higher rank lattices on one side and an arbitrary free ergodic action on the other side. Note however that Furman's theorem nevertheless depends on Zimmer's cocycle rigidity theorem. We also mention the orbit equivalence superrigidity theorems obtained by Monod and Shalom [39] for certain actions of direct products of hyperbolic groups. An excellent overview of orbit equivalence rigidity theory can be found in Shalom's survey [61].

Zimmer's cocycle rigidity theorem was a deep generalization of Margulis' seminal superrigidity theory [38]. In particular, the mathematics behind involve the theory of algebraic groups and their lattices. On the other hand, Popa's technique to deal with 1-cocycles for Bernoulli actions is intrinsically operator algebraic.

As stated above, Popa uses his powerful *deformation/rigidity strategy* to prove the cocycle superrigidity theorem. Leaving aside several delicate passages, the argument goes as follows. A 1-cocycle γ for the Bernoulli action $G \curvearrowright X$ of a w-rigid group G, can be interpreted in two ways as a 1-cocycle for the diagonal action $G \curvearrowright X \times X$, either as γ_1, only depending on the first variable, either as γ_2, only depending on the second variable. The malleability of the Bernoulli action (this is the deformation property) yields a continuous path joining γ_1 to γ_2. The relative property (T) implies that, in cohomology, the 1-cocycle remains essentially constant along the continuous path. This yields $\gamma_1 = \gamma_2$ in cohomology and the weak mixing property allows to conclude that γ is cohomologous to a homomorphism.

Let (σ_g) be the Bernoulli action of a w-rigid group G on (X, μ). Popa's cocycle superrigidity theorem covers his previous result [54, 57] identifying the 1-cohomology group $H^1(\sigma)$ with the character group $\operatorname{Char} G$. This result allows to compute as well the 1-cohomology for quotients of Bernoulli actions, yielding the following result of [53].

Theorem (Theorem 5.3). — *Let G be a w-rigid group. Then, G admits a continuous family of non-stably*[(3)] *orbit equivalent actions.*

Note that Popa does not only prove an existence result, but explicitly exhibits a continuous family of mutually non orbit equivalent actions. The existence of a continuum of non orbit equivalent actions of an infinite property (T) group had been established before in a non-constructive way by Hjorth [27], who exhibits a continuous family of actions such that every action in the family is orbit equivalent to at most countably many other actions of the family.

Finally note that the first concrete computations of 1-cohomology for ergodic group actions are due to Moore [40] and Gefter [23].

Von Neumann strong rigidity

The culmination of Popa's work on Bernoulli actions is the following *von Neumann strong rigidity* theorem of [56]; it is the first theorem in the literature that deduces conjugacy of the actions from isomorphism of the crossed product von Neumann algebras.

Theorem (Theorem 9.1). — *Let G be a group with infinite conjugacy classes and $G \curvearrowright (X, \mu)$ its Bernoulli action as above. Let Γ be a w-rigid group that acts freely and ergodically on (Y, η). If*

$$\theta : L^\infty(Y) \rtimes \Gamma \to p(L^\infty(X) \rtimes G)p$$

is a *-isomorphism for some projection $p \in L^\infty(X) \rtimes G$, then $p = 1$, the groups Γ and G are isomorphic and the actions of Γ and G are conjugate through this isomorphism.

Note that in the conditions of the theorem, there is an assumption on the action on one side and an assumption on the group on the other side. As such, it is not a superrigidity theorem: one would like to obtain the same conclusion for any free ergodic action of any group Γ and for the Bernoulli action of a w-rigid ICC group G.

Another type of von Neumann rigidity has been obtained by Popa in [52, 51], deducing orbit equivalence from von Neumann equivalence. We just state the following particular case. Consider the usual action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}^2$. Whenever a free and ergodic action of a group Γ with the Haagerup property is von Neumann equivalent with the $\mathrm{SL}(2, \mathbb{Z})$ action on $\mathbb{T}^2$, it actually is orbit equivalent with the latter. One should not hope to deduce a strong rigidity result yielding conjugacy of the actions: Monod and Shalom ([39], Theorem 2.27) proved that any free ergodic action of the

(3) See Definition 4.2.