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PROJECTIVITY OF KÄHLER MANIFOLDS – KODAIRA'S PROBLEM [after C. Voisin]

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There are various geometric structures that can be studied on a topological manifold M. Depending on one's geometric taste, it is important to know whether M can be endowed with a symplectic form, whether (special) Riemannian metrics can be found or whether M carries an algebraic structure. Often, the existence of a certain geometric structure imposes topological conditions on M. In other words, it may happen that a given topological manifold does simply not allow one's favorite geometry. *E.g.* if M is compact and $b_2(M) = 0$ the manifold M cannot be symplectic, or if $b_1(M) = 1$ no Kähler metrics can exist.

In order to fully understand the relation between two sorts of geometries, it is important to know whether they impose the same topological obstructions. In other words, does the existence of one of the two on a given manifold topological M imply the existence of the other one? This is a report on the work of Claire Voisin [13, 14] that sheds light on an old question, usually attributed to Kodaira, that asks for the topological relation between Kähler geometry and projective geometry.

In the following we let M be a compact manifold that can be endowed with the structure of a complex manifold. Once a complex structure is chosen, one studies Riemannian metrics g that are 'compatible' with it. One possible compatibility condition is to require that g be hermitian, *i.e.*, that the complex structure thought of as an almost complex structure I is orthogonal with respect to g. It is not difficult to see that a hermitian structure can always be found. It is, however, a completely different matter to find a hermitian structure g such that its fundamental form $\omega := g(I\ \ ,\ \ )$ is closed, *i.e.*, g satisfies the Kähler condition. Indeed, the classical theory of Kähler manifolds shows that the existence of a Kähler metric imposes strong conditions on the topology of M, which are not satisfied by arbitrary complex or symplectic manifolds. For instance, the odd Betti numbers of a compact Kähler manifold are even, Kähler manifolds are formal and their fundamental groups satisfy further conditions. (In contrast, if only one of the two structures, complex or symplectic, is required, then any finitely presentable group can be realized.)

On the other hand, Kähler manifolds are quite common. Indeed, any complex submanifold of the complex projective space $\mathbb{P}^n$ admits a Kähler metric - the restriction of the Fubini–Study metric is an example. Conversely, one might wonder whether a compact complex manifold that admits a Kähler structure can always be realized as a complex submanifold of $\mathbb{P}^n$ or, in other words, whether the complex structure is projective. This is obviously not the case, general complex tori $\mathbb{C}^n/\Gamma$ ($n \geq 2$) and general K3 surfaces provide counter-examples. In fact, a famous theorem of Kodaira proves that a Kähler manifold is projective if and only if the Kähler metric can be chosen such that the cohomology class of its fundamental form ω is integral, *i.e.*, $[\omega] \in H^2(X, \mathbb{Z})$ (see [6, Thm. 4]).

In these examples one observes that although the given complex structure is not projective, it becomes projective after a small deformation. Kodaira proved that in fact any Kähler surface can be deformed to a projective surface (see [7, Thm. 23] and [8]). Thus, as deforming the complex structure does not change the diffeomorphism type of the manifold, there is no topological difference between compact Kähler surfaces and algebraic surfaces. (Let us also mention that in fact any compact surface X with even $b_1(X)$ is Kähler, *i.e.*, for surfaces the condition to be Kähler is a topological condition. This fails in higher dimensions, due to a famous example of Hironaka [5] of a compact Kähler manifold that deforms to complex manifold which is no longer Kähler.) Note in passing that a similar result holds true for symplectic manifolds: clearly, any given symplectic form ω can be deformed to a symplectic form with integral cohomology class.

Kodaira's problem, which apparently has never been stated by himself in this form, asks for the higher-dimensional version of his result: *Can any compact Kähler manifold be deformed to a projective manifold?*

More in the spirit of the general philosophy explained above, one could ask whether the topological manifold underlying a compact Kähler manifold may also be endowed with the structure of a projective manifold. This question had been open for a very long time. As Kodaira's arguments to prove the two-dimensional case use a great deal of classification theory of surfaces, there was little hope to generalize them to higher dimensions.

Recent work of Claire Voisin fills this gap [11, 13, 14]. She succeeded in showing that topology makes a difference between compact Kähler manifolds and those that are projective. In other words, there exist compact topological manifolds that admit the structure of a Kähler manifold without carrying also the structure of a projective manifold. More precisely, Voisin shows the stronger statement:

THEOREM 0.1 ([13]). — *In any dimension ≥ 4 there exists a compact Kähler manifold X whose rational cohomology ring $H^*(X, \mathbb{Q})$ cannot be realized as the rational cohomology ring of a projective manifold.*

Voisin originally worked with the integral cohomology ring $H^*(X,\mathbb{Z})$, but Deligne then pointed out the stronger version above.

One could wonder whether the answer to these questions would be different if the topological manifold satisfies further conditions, *e.g.* if it is in addition simply-connected. Some of these questions have been addressed and answered by Voisin in [13, 14] and we will comment on them on the way.

Although the examples are obtained by particular constructions, the principal ideas of [13, 14] are of a more general nature and might be applicable in other situations.

The i-th cohomology of a compact Kähler manifold is naturally endowed with a Hodge structure of weight i, which can be polarized (on the primitive part) if the manifold is projective. The idea is to show that there exist compact Kähler manifolds whose cohomology does not admit Hodge structures that are compatible with both, the given cup-product and a polarization. Roughly, there are three steps **A-C**, the first two of which are purely Hodge-theoretical and only the last one has a geometric flavor.

(A) Certain algebraic structures on a rational vector space A are not compatible with any *polarizable* Hodge structure (of weight k) on A.

Remark 0.2. — In the examples, the algebraic structure will be a specific endomorphism $\Phi : A \to A$, but others are in principle possible. That the algebraic structure is not compatible with any polarizable Hodge structure means in the case of an endomorphism Φ that one cannot find a Hodge structure on A such that Φ becomes an endomorphism of it and such that the Hodge structure can be polarized.

(B) Suppose $\bigoplus H^\ell$ is a graded $\mathbb{Q}$-algebra whose direct summands H^ℓ are Hodge structures of weight ℓ and such that the multiplications $H^{\ell_1} \otimes H^{\ell_2} \to H^{\ell_1+\ell_2}$ are homomorphisms of Hodge structures. Suppose furthermore that this $\mathbb{Q}$-algebra structure allows us to detect a subspace $A \subset H^k$ such that: i) $A \subset H^k$ is a Hodge substructure. ii) An algebraic structure as in **(A)** is compatible with this Hodge structure.
Then H^k does not admit a polarization.

Remark 0.3. — Subspaces that are defined purely in terms of the $\mathbb{Q}$-algebra structure do define Hodge substructures. We shall also need a refined version of this, which is due to Deligne.

The compatibility in ii) is more difficult to check, but relies on the same principle. For an endomorphism Φ the idea goes as follows: Firstly, find two Hodge substructures $A, A' \subset H^k$ and a Hodge substructure $\Delta \subset A \oplus A' \subset H^k$ which is the graph of an isomorphism $A \cong A'$. Secondly, prove that under the induced isomorphism of Hodge structures $A \oplus A \cong A \oplus A'$ the graph of Φ is a Hodge substructure.

(C) Construct compact Kähler manifolds such that the above principles apply to its cohomology ring $\bigoplus H^\ell(X,\mathbb{Q})$. Then $H^*(X,\mathbb{Q})$ should not be realizable by a smooth projective variety.

Remark 0.4. — This works best for Hodge structures of weight one ($k=1$). In this case $H^1(X,\mathbb{Q})$ of a smooth projective variety X admits a polarized Hodge structure. For the Hodge structure of weight two on $H^2(X,\mathbb{Q})$ one needs an extra argument, for only the primitive part of it admits a polarization.

This report roughly follows these three steps. Some of the algebraic structures in Section 2 might seem rather ad hoc, as their geometric origin is only explained in Section 3. However, I found it helpful for my own understanding to completely separate the arguments that explain why certain $\mathbb{Q}$-algebras cannot be realized as the cohomology of a projective manifold from the part that contains the construction of compact Kähler manifolds that do realize these $\mathbb{Q}$-algebras.

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1. HODGE STRUCTURES (OF WEIGHT ONE AND TWO)

1.1. Recollections

A *Hodge structure of weight* k on a $\mathbb{Q}$-vector space A is given by a direct sum decomposition

$$A_{\mathbb{C}} := A \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} A^{p,q} \quad \text{such that} \quad \overline{A^{p,q}} = A^{q,p}. \tag{1}$$

A direct sum decomposition (1) can also be described in terms of a representation $\rho : \mathbb{C}^* \to \mathrm{Gl}(A_{\mathbb{R}})$ such that the $\mathbb{C}$-linear extension of $\rho(z)$ satisfies $\rho(z)|_{A^{p,q}} = z^p \bar{z}^q \cdot \mathrm{id}$. The *Hodge classes* of a Hodge structure of weight $2k$ on A are the elements in $A^{k,k} \cap A$.

We shall be particularly interested in Hodge structures of weight one and two.

Remark 1.1. — Recall that Hodge structures of weight one with $A^{p,q}=0$ for $pq \neq 0$ which are integral, *i.e.*, $A = \Gamma_{\mathbb{Q}}$ for some lattice Γ, are in bijection with complex tori. Indeed, to a Hodge structure of weight one on $\Gamma_{\mathbb{Q}}$ given by $\Gamma_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ one associates the complex torus $A^{1,0}/\Gamma$, where Γ is identified with its image under the projection $A_{\mathbb{C}} \to A^{1,0}$.

A $\mathbb{Q}$-linear map $\varphi : A \to A'$ is a *morphism* (*of weight* m) *of Hodge structures*

$$A_{\mathbb{C}} = \bigoplus_{p+q=k} A^{p,q} \quad \text{and} \quad A'_{\mathbb{C}} = \bigoplus_{r+s=\ell} A'^{r,s}$$

of weight k and $\ell = k + 2m$, respectively, if $\varphi(A^{p,q}) \subset A'^{p+m,q+m}$. If the two Hodge structures correspond to $\rho : \mathbb{C}^* \to \mathrm{Gl}(A_{\mathbb{R}})$ and $\rho' : \mathbb{C}^* \to \mathrm{Gl}(A'_{\mathbb{R}})$, respectively, then this condition is equivalently expressed by $\varphi(\rho(z)v) = |z|^{2m}\rho'(z)\varphi(v)$ for all $v \in A$ and $z \in \mathbb{C}^*$.

A *Hodge substructure* of a Hodge structure of weight k on A is given by a subspace $A' \subset A$ such that $A'_{\mathbb{C}} = \bigoplus(A^{p,q} \cap A'_{\mathbb{C}})$ or, equivalently, such that $A'_{\mathbb{C}} \subset A_{\mathbb{C}}$ is invariant under the representation $\rho : \mathbb{C}^* \to \mathrm{Gl}(A_{\mathbb{R}})$ that corresponds to the given Hodge structure on A.

The *tensor product* $A \otimes_{\mathbb{Q}} A'$ of two $\mathbb{Q}$-vector spaces A and A' endowed with Hodge structures of weight k and ℓ, respectively, comes with a natural Hodge structure of weight $(k + \ell)$:

$$(A \otimes_{\mathbb{Q}} A')^{r,s} := \bigoplus_{p+p'=r, q+q'=s} A^{p,q} \otimes_{\mathbb{C}} A'^{p',q'}.$$

In other words, the Hodge structure is given by $\rho \otimes \rho'$.

Note that $A_2 := \bigwedge^2 A_1$ of a Hodge structure of weight one A_1 is naturally a Hodge structure of weight two with $A_2^{2,0} := \bigwedge^2 A_1^{1,0}$, $A_2^{1,1} := A_1^{1,0} \otimes A_1^{0,1}$, and $A_2^{0,2} := \bigwedge^2 A_1^{0,1}$.

A *polarization* of a Hodge structure of weight one $A_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ is a skew-symmetric form $q \in \bigwedge^2 A^*$ such that

$$A_{\mathbb{C}} \times A_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad (v, w) \longmapsto iq(v, \overline{w}) \tag{2}$$

(where q is extended $\mathbb{C}$-linearly) satisfies the Hodge–Riemann relations:

i) $A^{1,0}$ and $A^{0,1}$ are orthogonal with respect to (2).

ii) The restriction of (2) to $A^{1,0}$ and to $A^{0,1}$ is positive, respectively negative, definite.

Remark 1.2. — With this definition a polarization is always rational. Furthermore, the form q considered as an element of the induced weight-two Hodge structure on $\bigwedge^2 A^*$ is of type $(1, 1)$. Since it is rational, q is a Hodge class (of weight two). Note that any Hodge substructure of a weight-one polarized Hodge structure is naturally polarized.

Example 1.3. — Let X be a compact Kähler manifold of dimension n. The Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

defines a Hodge structure of weight one on $H^1(X, \mathbb{Q})$.

Suppose X is projective and $\omega \in H^2(X, \mathbb{Z})$ is the class of a hyperplane section, then $q(\alpha) = \int_X \alpha^2 \omega^{n-1}$ is a polarization of the natural Hodge structure of weight one on $H^1(X, \mathbb{Q})$.