326

ASTÉRISQUE

2009

SÉMINAIRE BOURBAKI
VOLUME 2007/2008
EXPOSÉS 982-996

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Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Séminaire BOURBAKI Novembre 2007
60e année, 2007–2008, n° 986, p. 143 à 164

RAMANUJAN'S MOCK THETA FUNCTIONS AND THEIR APPLICATIONS [d'après Zwegers and Ono–Bringmann]

by **Don ZAGIER**

INTRODUCTION

One of the most romantic stories in the history of mathematics is that of the friendship between Hardy and Ramanujan. It began and ended with two famous letters. The first, sent by Ramanujan to Hardy in 1913, presents its author as a penniless clerk in a Madras shipping office who has made some discoveries that "are termed by the local mathematicians as 'startling'." Hardy spent the night with Littlewood convincing himself that the letter was the work of a genius and not of a fraud and promptly invited Ramanujan to come to England for what was to become one of the most famous mathematical collaborations in history. The other letter was sent in 1920, also by Ramanujan to Hardy, just three months before his death at the age of 32 in India, to which he had returned after five years in England. Here he recovers briefly from his illness and depression to tell Hardy excitedly about a new class of functions that he has discovered and that he calls "mock theta functions."

This letter has become celebrated, not only because of the tragic circumstances surrounding it, but also because it was mathematically so mysterious and intriguing. Ramanujan gives no definition of mock theta functions but only a list of 17 examples and a qualitative description of the key property that he had noticed: that these functions have asymptotic expansions at every rational point of the same type as those of theta functions (Ramanujan used the word "theta functions" where we would say "modular forms" today, so that "mock theta functions" meant something like "fake modular forms"), but that there is no single theta function whose asymptotic expansion agrees at all rational points with that of the mock theta function. Obviously, this is a basic property, but far from a complete definition.

In the years since 1920, many papers have been written, including many by famous mathematicians like Watson, Selberg and Andrews, studying the 17 specific examples Ramanujan had given, proving the identities that he had stated, and finding further identities of the same type. But no natural definition was known that described what these functions are intrinsically and hence could give a natural explanation of the identities between them and a method to construct further examples at will. The breakthrough came in 2002 with the thesis of a Dutch doctoral student, Sander Zwegers, who finally found the missing intrinsic characterization of mock theta functions. In fact, he did this in three different ways! Specifically, he observed that various known identities from the literature could be interpreted as saying that each of Ramanujan's examples belongs to at least one (and presumably to all, although probably not all 51 verifications have been carried out explicitly) of three infinite families of functions:

(A) "Lerch sums"

(B) "Quotients of indefinite binary theta series by unary theta series"

(C) "Fourier coefficients of meromorphic Jacobi forms"

(We will define and discuss these families in more detail below.) For each of these classes he was able to prove a specific type of near-modular behavior which therefore held in particular for Ramanujan's examples. What's more, this near-modularity property turned out to be the same for each of the three classes, so that the original problem was not only triply solved, but in a way that made it quite convincing that the essential property of these functions really had been correctly identified.

In this talk we will describe Ramanujan's letter and the 17 original examples, describe each of the classes (A) – (C) and the nature of their modularity, formulate a general definition of mock modular forms, and describe further examples. In the final section, we will also discuss some of the beautiful recent work of Kathrin Bringmann and Ken Ono, based on these ideas, that has led to the solution of several well-known open problems in combinatorics and the theory of q-series.

Before beginning the main story, there are two points that I would like to emphasize. The first is that one of the reasons for the great usefulness (or "unreasonable effectiveness," to coin a phrase) of classical modular forms in number theory is that each modular form has calculable invariants—its weight, level, and a (known) finite number of its first Fourier coefficients—that suffice to characterize it uniquely. This means that to prove any conjectured identity between modular forms, like the famous formulas

$$q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=1}^{\infty}\Big(\frac{12}{n}\Big)q^{n^2/24}, \qquad \Big(\sum_{n\in\mathbb{Z}}(-q)^{n^2}\Big)^4+\Big(\sum_{n\in\mathbb{Z}+\frac12}q^{n^2}\Big)^4 = \Big(\sum_{n\in\mathbb{Z}}q^{n^2}\Big)^4$$

of Euler and Jacobi, respectively, it suffices to calculate the invariants on both sides and check that they are the same; one does not need to know any further properties of the functions involved or even where they come from. Precisely the same principle will apply also to the larger class of mock modular forms motivated by and containing

Ramanujan's examples, as soon as we know their modular transformation properties, so that here, too, identities which previously required lengthy computations and great ingenuity for their proofs can now be established by an essentially automatic procedure.

The second point is that all 17 of Ramanujan's mock theta functions were given in the form of q-hypergeometric series. (We recall that a q-hypergeometric series is a sum of the form $\sum_{n=0}^{\infty} A_n(q)$ where each $A_n(q) \in \mathbb{Q}(q)$ and $A_{n+1}(q)/A_n(q) = R(q,q^n)$ for all $n \geq 1$ for some fixed rational function $R(q,r) \in \mathbb{Q}(q,r)$.) Some modular forms are q-hypergeometric series, classical examples being the theta series and Eisenstein series

$$\frac{1}{2} + \sum_{n=1}^{\infty} q^{n^2} \qquad \text{and} \qquad \frac{1}{24} + \sum_{n=1}^{\infty} q^n \, \frac{1+q^{2n}}{(1-q^{2n})^2} \, ,$$

respectively, but this is very rare and there is no known criterion for deciding whether a given q-hypergeometric series is modular or not. (There are fascinating conjectures due to Werner Nahm relating this question to deep questions of conformal field theory and algebraic K-theory [**18, 25**].) Ramanujan loved and was a supreme connoisseur of q-hypergeometric series, and his examples all quite naturally belonged to this category, but it is a complete red herring from the point of view of understanding the intrinsic modular transformation properties that make these functions special. It is perhaps precisely for this reason that it took so long for these transformation properties to be found, just as the theory of ordinary modular forms would have developed much more slowly if for some reason one had focused only on the rare q-hypergeometric examples.

1. RAMANUJAN'S LETTER

Ramanujan divided his seventeen examples into four of order 3, ten of order 5, and three of order 7, though he gave no indication what these "orders" were. (We'll see later that they are related to the levels of the corresponding mock modular forms.) We will discuss most of these functions here to illustrate various points involved.

The mock theta functions of order 3 were denoted f, ϕ, ψ and χ. We give only the first three (changing q to $-q$ in ϕ and ψ in order to simplify the relations):

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2} \, ,$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} \, ,$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})} .$$

Ramanujan gives two relations among these functions (as well as a further relation involving f and χ), all proved later by Watson:

$$2\phi(q) \;-\; f(q) \;=\; f(q) \;+\; 4\psi(q) \;=\; \frac{1-2q+2q^4-2q^9+\cdots}{(1+q)(1+q^2)(1+q^3)\cdots},$$

where the expression on the right-hand side is, up to a factor $q^{-1/24}$, a modular form of weight $\frac{1}{2}$. Already in this first example we see three points:

- there are linear relations among the mock theta functions (here, $\phi = f + 2\psi$);
- the space they span contains a subspace of ordinary modular forms;
- one must multiply by suitable powers of q to get the correct modular behavior.

Ramanujan also describes the asymptotics of $f(q)$ as q tends to any root of unity, a typical result being

$$e^{\pi t/24}\, f\big(-e^{-\pi t}\big) \;=\; -\frac{1}{\sqrt{t}}\, e^{\pi/24t} \;+\; 4 \;+\; \mathrm{o}(1) \qquad \text{as } t \to 0\,.$$

Notice that, as Ramanujan asserts, this is similar to the type of expansion which we would have if $q^{-1/24} f(q)$ were a true modular form of weight $\frac{1}{2}$, except that then the subleading terms would have a form like $t^{-1/2}\sum_{n\ge 0} a_n\, e^{-\pi n/24t}$ rather than $4+\mathrm{o}(1)$.

The ten mock theta functions of order 5 have similar features, but are considerably more complicated. We discuss this case in more detail since it is quite typical. The functions come in five groups of two each, denoted f_j, ϕ_j, ψ_j, χ_j and F_j with $j \in \{1,2\}$. (These are Ramanujan's notations, except that he omits the indices.) The five functions with index $j = 1$ are given by

$$\begin{aligned}
f_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\,\cdots(1+q^n)}\,,\\
\phi_1(q) &= \sum_{n=0}^{\infty} q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1})\,,\\
\psi_1(q) &= \sum_{n=1}^{\infty} q^{n(n+1)/2}\,(1+q)(1+q^2)\cdots(1+q^{n-1})\,,\\
\chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})\cdots(1-q^{2n})}\,,\\
F_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}\,,
\end{aligned}$$