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(989) *The strong abc conjecture over function fields*

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THE STRONG *abc* CONJECTURE OVER FUNCTION FIELDS [after McQuillan and Yamanoi]

by **Carlo GASBARRI**

1. INTRODUCTION

One of the deepest conjectures in arithmetics is the *abc* conjecture:

Conjecture 1.1. — *Let $\epsilon > 0$; then there exists a constant $C(\epsilon)$ for which the following holds: Let a, b and c be three integral numbers such that $(a,b) = 1$ and $a + b = c$. Then*

$$\max\{|a|,|b|,|c|\} \leq C(\epsilon)\Big(\prod_{p/abc} p\Big)^{1+\epsilon},$$

where the product is taken over all the prime numbers dividing abc.

Let us give a geometric interpretation of this conjecture:

Consider the arithmetic surface $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ equipped with the tautological line bundle $\mathcal{O}(1)$ and the divisor $D := [0:1]+[1:0]+[1:-1]$. Suppose we have a section $P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$, not contained in D; then $P^*(D)$ is an effective Weil divisor on $\mathrm{Spec}(\mathbb{Z})$ which can be written as $\sum_p v_p(D)[p]$.

Define *the radical* of the divisor as $N^{(1)}_D(P) := \sum_p \min(1, v_p(D))\log(p)$.

The conjecture can be stated in this way: for every $\epsilon > 0$, there is a constant $C(\epsilon)$ such that, for every section $P : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$, we have

$$h_{\mathcal{O}(1)}(P) \leq (1+\epsilon)N^{(1)}_D(P) + C(\epsilon),$$

where $h_{\mathcal{O}(1)}(P)$ is the height of P with respect to $\mathcal{O}_{\mathbb{P}^1}(1)$. When we state the conjecture in this way we see many possible generalizations. We also clearly see the geometric analogue over function fields (cf. next sections for details). Let us formulate the conjecture in the most general version.

If K is a number field, we denote by $\mathcal{O}_K$ the ring of integers of K and by Δ_K its discriminant. If $X \to K$ is an arithmetic surface, D an effective divisor over X and

$P : \mathrm{Spec}(\mathcal{O}_{\mathrm{K}}) \to \mathrm{X}$ a section not contained in D, we define the *radical* of D as the real number

$$N_D^{(1)}(P) := \sum_{\mathfrak{p}\in \mathrm{Spec\,max}(\mathcal{O}_{\mathrm{K}})} \min\{1; v_{\mathfrak{p}} P^*(D)\} \log \mathrm{Card}(\mathcal{O}_K/\mathfrak{p}).$$

The general strong *abc* conjecture is the following:

CONJECTURE 1.2. — *Let $\epsilon > 0$, and let K be a number field, $\pi : X \to \mathrm{Spec}(\mathcal{O}_{\mathrm{K}})$ a regular arithmetic surface and $D \hookrightarrow X$ an effective divisor on X. Denote by $K_{X/\mathcal{O}_K}$ the relative dualizing sheaf. Then there exists a constant $C := C(X, \epsilon, D)$ for which the following holds: Let L be a finite extension of K and let $P : \mathrm{Spec}(\mathcal{O}_{\mathrm{L}}) \to \mathrm{X}$ be a section not contained in D; then*

$$h_{K_{X/\mathcal{O}_K}(D)}(P) \leq (1+\epsilon)(N_D^{(1)}(P) + \log|\Delta_L|) + C[L : K],$$

where $h_{K_{X/\mathcal{O}_K}(D)}(P)$ is the height of P with respect to $K_{X/\mathcal{O}_K}(D)$.

We will not list here the endless number of consequences of this conjecture and we refer to [**3**] or to the web page [**19**] for details. One may also see the report [**21**] in this seminar. We only notice that, if such a conjecture was true, more or less all the possible problems about the arithmetic of algebraic curves over number fields would have an effective answer: for instance one easily sees that, if the constant C is effective, it easily implies the famous Fermat Last Conjecture (now a theorem [**28**]) and it allows to solve effectively diophantine equations in two variables:

THEOREM 1.3. — *Suppose that Conjecture 1.2 is true. Let $F(x; y) \in \mathbb{Z}[x, y]$ be an irreducible polynomial of degree at least three. Then there exists a constant C, depending only on F, such that for every number field K and for every solution $(x; y) \in \mathcal{O}_K \times \mathcal{O}_K$ of the diophantine equation $F(x; y) = 0$, we have*

$$h_{\mathcal{O}(1)}([x : y : 1]) \leq (1+\epsilon)\log|\Delta_K| + C_\epsilon[K : \mathbb{Q}].$$

In particular there are only finitely many solutions in $\mathcal{O}_K \times \mathcal{O}_K$ and their height can be explicitly bounded.

Observe that, if the conjecture is true and the constant C_ϵ is explicit, then we can explicitly *compute and find* the set of solutions of the diophantine equation in $\mathcal{O}_K \times \mathcal{O}_K$.

Similarly we may obtain an effective version of Mordell's conjecture (Faltings' theorem) and of the classical Siegel theorem on integral points of hyperbolic curves.

At the moment we know that the set of integral points of a hyperbolic curve is finite (projective by Faltings' theorem [**7**] or affine by Siegel's theorem, cf. [**24**]) but we are not able to explicitly bound their height (up to some sporadic cases); thus, in particular, it is not possible to find all the rational points of a hyperbolic curve.

In this paper we will report about the solution of the analogue of the *abc* conjecture over function fields (for the analogy between number fields and function fields arithmetic cf. for instance [**24**]).

The analogue of Conjecture 1.1 for polynomials is quite easy and proved in [**12**]: If f is a polynomial over $\mathbb{C}$ (to simplify), let $N_0 = (f)$ be the number of *distinct* roots of f. Then the analogue of the *abc* conjecture for polynomials is

Theorem 1.4 (Mason). — *Let f, g and h be three polynomials relatively coprime in $\mathbb{C}[t]$ such that $f+g=h$; then*

$$\max\{\deg(f), \deg(g), \deg(h)\} \leq N_0(fgh) - 1.$$

This theorem is the analogue of Conjecture 1.2 for function fields when $X = \mathbb{P}^1 \times \mathbb{P}^1$, $\pi : X \to \mathbb{P}^1$ is the first projection, $D = \mathbb{P}^1 \times [0:1] + \mathbb{P}^1 \times [1:0] + \mathbb{P}^1 \times [1:-1]$ and P is a section. One easily deduces it from Hurwitz's formula (cf. next section). It can be seen as the beginning of all the story, and it has some interesting consequences: for example the analogue of Fermat's last theorem for polynomials is an immediate consequence of it. Usually statements in the function fields situation are much easier to prove than their correspondent in the number fields situation. In this case one should notice an amazing point: Suppose that, over number fields, we can prove Conjecture 1.2 when $X = \mathbb{P}^1_{\mathbb{Z}}$ and $D = [0:1] + [1:0] + [1:-1]$; then we can deduce the general case from this! To prove it, one applies the proof of Theorem 7.1 to a suitable Belyi map (for more details, see [**6**]). In the function fields case this is *not* the case! We cannot deduce the general case from an isotrivial case. For this reason it is our opinion that $\mathbb{P}^1_{\mathbb{Z}}$ with the divisor $[0:1] + [1:0] + [1:-1]$ (unit equations) is a highly non isotrivial family over $\mathrm{Spec}(\mathbb{Z})$ (whatever an isotrivial family should be).

Exploiting the analogy between the arithmetic geometry over number fields and the theory of *analytic* maps from a parabolic curve to a surface (cf. for instance [**26**]), an analogue of the *abc* conjecture for these maps is also solved.

We will propose two proofs of the *abc* conjecture over function fields (and for analytic maps). The first one is the proof by McQuillan [**15**] and the second one is by Yamanoi [**29**]. The proof by McQuillan is synthetically explained in the original paper; it makes a systematic use of the theory of integration on algebraic stacks; although this is very natural in this context, it needs a very heavy background (which is used here only in a quite easy situation). Thus we preferred to propose a self contained proof which uses the (easier) theory of normal $\mathbb{Q}$–factorial varieties; the proof follows the main ideas of the original one. The proof by Yamanoi requires skillful combinatorial computations, well explained in the original paper, thus we preferred to sketch his proof in a special (but non trivial) case: the main ideas and tools are all used and we think that once one understands this case, it is easier to follow the proof in the general situation.

As before, we find for instance, as a consequence, a strong effective version of Mordell's conjecture over function fields (in characteristic zero), for non isotrivial families of hyperbolic curves.

In the next section, we will explain why the *abc* conjecture for isotrivial curves corresponds respectively to the Hurwitz formula in the geometric case and to the Nevanlinna Second Main theorem in the analytic case. Thus the *abc* conjecture may be seen as a non isotrivial version of these theorems.

There are at least two strategies to attack the Second Main Theorem of Nevanlinna's theory. The first strategy uses tools from analytic and differential geometry, it is strictly related to the algebraic geometry of the Hurwitz formula and to the existence of particular singular metrics on suitable line bundles: it has been strongly generalized to analytic maps between equidimensional varieties by Griffiths, King and others in the 70's (cf. [**8**]). The second strategy is via Ahlfors' theory (cf. [**1**]); it is much related to the algebraic and combinatorial topology of maps between surfaces; the version of the SMT one obtains in this way is weaker than the original one but also more subtle: one sees that one can perturb a little bit the divisor D without perturbing the statement (cf. § 8). These two approaches correspond respectively to the two proposed proofs. The proof by McQuillan is nearer to the first strategy while the one by Yamanoi is more topological. One should notice that, while the first proof is predominantly of a global nature and the second one is essentially local, both meet the main difficulties in an argument which is localized around the singular points of the morphism $p : X \to B$. If the morphism p is relatively smooth, McQuillan's proof is much simpler. In a hypothetical relatively smooth case, Yamanoi's approach reduces to the Ahlfors theory: you will observe that, unless you are in the isotrivial case, in Yamanoi's approach there is always a bad reduction.

Both proofs hold for curves over function fields in one variable over $\mathbb{C}$ and both heavily use analytic and topological methods, specific of the complex topology. We should notice that the analogue of the *abc* conjecture, as stated before, over a function field with positive characteristic is false (cf. [**11**])!

1.1. A short overview of the history of the *abc* conjecture

The *abc* conjecture has a weak and a strong version (in the arithmetic case they are both unproven and very deep). Over function fields, the weak *abc* is easier to prove and it is strictly related with the theory of elliptic curves (cf. [**10**] and [**25**]). Here we deal with the strong version. The conjecture has been formulated in the middle 80's by Masser and Oesterlé exploiting the analogy between number fields and function fields and the version for polynomials proved in [**12**]. The general version,