363-364

ASTÉRISQUE

2015

SÉMINAIRE BOURBAKI
VOLUME 2013/2014
EXPOSÉ N° 1082

Erwin BOLTHAUSEN

Ultrametricity in mean-field spin glasses

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 367-368

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 90 € ($ 135)
Abonnement Europe : 650 €, hors Europe : 689 € ($ 1 033)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0303-1179

ISBN 978-2-85629-804-6

Directeur de la publication : Marc PEIGNÉ

Séminaire BOURBAKI Mars 2014
66e année, 2013-2014, n° 1082, p. 255 à 283

ULTRAMETRICITY IN MEAN-FIELD SPIN GLASSES

by **Erwin BOLTHAUSEN**

1. MEAN-FIELD SPIN GLASSES

Spin glasses are Gibbs measures with random interactions. The most natural one is probably the Edwards-Anderson model, which is an Ising type model with nearest neighbor random interactions. On a finite box $V_N \stackrel{\text{def}}{=} \{1, \ldots, N\}^d$ with outer boundary ∂V_N (consisting in the points in $\mathbb{Z}^d \backslash V_N$ which have a neighbor in V_N) one defines the Hamiltonian with boundary condition $\eta \in \{-1, 1\}^{\partial V_N}$ by

$$-H_{V_N}(\sigma|\eta) \stackrel{\text{def}}{=} \sum_{i,j \in V_N,\ i\sim j} J_{ij}\sigma_i\sigma_j + \sum_{i \in V_N, j \in \partial V_N, i\sim j} J_{ij}\sigma_i\eta_j,$$

for $\sigma \in \{-1, 1\}^{V_N}$, where $i \sim j$ means that the points are neighbors on the lattice. One always takes $J_{ij} = J_{ji}$, so that the sum is over the undirected bonds of the graph V_N. In the Ising model, one would have $J_{ij} = J = \text{const} > 0$. The Hamiltonian defines a Gibbs measure on $\Sigma_{V_N} \stackrel{\text{def}}{=} \{-1, 1\}^{V_N}$ with boundary condition η, and with inverse temperature parameter $\beta > 0$, by

$$\mu_{V_N,\eta}(\sigma) \stackrel{\text{def}}{=} \frac{1}{Z_{N,\eta,\beta}} \exp\left[-\beta H_{V_N}(\sigma|\eta)\right].$$

The normalizing constant $Z_{N,\eta,\beta}$, the so-called partition function (as it is a function of β) is

$$Z_{N,\eta,\beta} \stackrel{\text{def}}{=} \sum_{\sigma \in \Sigma_{V_N}} \exp\left[-\beta H_{V_N}(\sigma|\eta)\right].$$

A natural question is about the possible limits with $N \to \infty$ and a sequence $\{\eta_N\}$ of boundary conditions. In particular, one is interested to know whether the set of measures on $\{-1, 1\}^{\mathbb{Z}^d}$ which can be obtained as limits contains a unique element, and if not how many extreme points in this set are. (The set is convex.)

For the Ising model, this is a well studied question, still with many open questions particularly for $d \geq 3$. The Edwards-Anderson model takes the J_{ij} as i.i.d. random variables, for instance centered Gaussian ones, or ± 1 coin tossings. The Hamiltonian itself is then a random variable, and the Gibbs measure is a random measure: If the J's are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the partition function is a function of $\omega \in \Omega$: $Z_{N,\eta,\beta,\omega}$, and the Gibbs measure also depends on the realizations ω. Typically, one is interested in so-called quenched properties, i.e., properties which hold for almost all ω (under $\mathbb{P}$). Unfortunately, for the Edwards-Anderson model, the most basic questions are mathematically completely open.

Mean-field models offer a possibility to investigate some of the basic questions, like the existence of phase transitions, in a mathematically much simpler way than for short range models like the Ising model. In mean-field models, the interactions are not local or short range, but a spin variable σ_i interacts with the others in a more global way, for instance through their means. However, the notion of an infinite Gibbs measure typically makes no longer sense, but one can still define the limiting free energy, mean magnetization, critical exponents, etc. Of course, in many respects, mean-field models are too simple to give even qualitatively the same answers as in short range models, for instance about critical exponents.

Given the mathematical difficulty in understanding short range spin glasses, it is natural to investigate mean-field type spin glasses, in the hope that they are tractable. This was the motivation of Sherrington and Kirkpatrick [**22**] to propose their now famous model. In the end, it turned out that a mathematically rigorous understanding is possible, in contrast to the present situation for short range models, but it took a long way, and the results are still far from covering all aspects. Here is the SK-model:

One starts with a countable number of standard Gaussian random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed as a matrix $(g_{ij})_{1\leq i<j}$. If we specify the dependence on ω, we write $g_{ij}(\omega)$. Of course, one can take $\Omega = \mathbb{R}^{\mathbb{N}}$, but probabilists usually don't like to fix that. Then for any $N \in \mathbb{N}$, one defines the random Hamiltonian

$$H_{N,\omega}(\sigma) \stackrel{\text{def}}{=} -\frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}(\omega)\,\sigma_i\sigma_j, \tag{1}$$

where $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N \stackrel{\text{def}}{=} \{-1,1\}^N$. The minus sign is of course of no relevance, and is put only to please the physicists. A slight generalization is to allow the presence of an external field in the form

$$H_{N,\omega}(\sigma) \stackrel{\text{def}}{=} -\frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}(\omega)\,\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \tag{2}$$

with $h \in \mathbb{R}$ a parameter. The external field adds some non-trivial complications. We will however stick to the Hamiltonian (1). As in the usual mean-field models, there is no geometric structure of the index set $\{1, \dots, N\}$.

From standard mean-field models one would expect a factor $\frac{1}{N}$ instead of $\frac{1}{\sqrt{N}}$. However, a moment's reflection shows that the square root is the appropriate scaling: The main issue is that, given a site i, the influence of the other sites on σ_i is of order 1. To achieve this in the random situation, one has to have the factor $\frac{1}{\sqrt{N}}$ in front.

One then defines in the usual way the partition function which now depends on the Gaussian variables g_{ij}, i.e., on ω:

$$Z_{N,\beta,\omega} \stackrel{\text{def}}{=} \sum_{\sigma} \exp\left[-\beta H_{N,\omega}(\sigma)\right], \tag{3}$$

and the random Gibbs measure

$$G_{N,\beta,\omega}(\sigma) \stackrel{\text{def}}{=} \frac{1}{Z_{N,\beta,\omega}} \exp\left[-\beta H_{N,\omega}(\sigma)\right]. \tag{4}$$

There are two sources of probability, namely for fixed ω the probability measure $G_{N,\beta,\omega}$ on Σ_N, and then the randomness of this law itself, as it depends in a non-trivial way on ω. One usually calls this a random probability distribution.

Remark 1. — *For the sake of simplicity, we restrict the discussion here completely to the original SK-model. Everything works for so-called p-spin models and mixtures of p-spin models. The p-spin models have the Hamiltonian*

$$H_N^{(p)}(\sigma) \stackrel{\text{def}}{=} \frac{1}{N^{(p-1)/2}} \sum_{1 \le i_1 < i_2 < \cdots < i_p \le N} g_{i_1, i_2, \dots, i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p},$$

where the g's as before are i.i.d. standard Gaussians.

We can now formulate the ultrametricity conjecture, although its importance will stay quite mysterious for the moment (and maybe still till the end of these notes).

A metric d on a space S is called an *ultrametric*, if it satisfies the stronger triangle inequality:

$$d(x, y) \le \max\left(d(x, z), d(y, z)\right), \ \forall x, y, z \in S.$$

This is essentially equivalent with the metric space having a tree structure, S being the set of leaves, and d being the (weighted) graph distance. Σ_N is a metric space under the Hamming distance, counting the number of sites on which two elements differ. This is evidently not an ultrametric. However, the ultrametricity conjecture states that it is approximately so under the above Gibbs measure, as $N \to \infty$.