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THE BOUNDED L^2 CURVATURE CONJECTURE
[after S. Klainerman, I. Rodnianski and J. Szeftel]

by Jacques SMULEVICI

1. INTRODUCTION

General relativity is one of the pillars of modern theoretical physics. Mathematically, it consists in the study of Lorentzian manifolds, typically in dimension $3 + 1$, satisfying the so-called *Einstein equations*, a system of geometric partial differential equations for the components of the Lorentzian metric of the manifold. In the absence of any matter source⁽¹⁾, they take the form

$$(1) \quad \text{Ric}(g) = 0,$$

where $\text{Ric}(g)$ denotes the Ricci tensor⁽²⁾ of a Lorentzian metric g .

As was understood already by Einstein himself, equations (1) are of wave type so that, being in particular evolution equations, the natural problem associated to them is the Cauchy problem. The initial data, formulated geometrically, consist in a triplet (Σ, h, k) where Σ is a manifold of dimension⁽³⁾ 3, h is a Riemannian metric on Σ , k is a symmetric 2-tensor and the following system of constraint equations holds

$$(2) \quad R^{(h)} - |k|^2 + (\text{tr}_h k)^2 = 0,$$

$$(3) \quad \text{div}_h k - \nabla^{(h)} \text{tr}_h k = 0,$$

where $R^{(h)}$ is the scalar curvature of h , $\nabla^{(h)}$ is the Levi-Civita connection associated to h , $\text{div}_h k$ is the divergence of k , and $\text{tr}_h k := k_{ab} h^{ab}$ denotes the trace of the 2-tensor k .

⁽¹⁾ They are then typically called the Einstein *vacuum* equations.

⁽²⁾ For the reader unfamiliar with some of notions in geometry needed to read this text, certain basic definitions of Lorentzian and Riemannian geometry are given at the beginning of Section 2.

⁽³⁾ The equations can naturally be posed in higher or lower dimensions, but in this text, we shall consider only the $3 + 1$ dimensional case.

A solution to the initial value problem associated to (Σ, h, k) , which we shall call a development of the corresponding data, is then a Lorentzian manifold (\mathcal{M}, g) of dimension $3 + 1$ satisfying the Einstein equations (1) and such that there exists an embedding of Σ into \mathcal{M} with (h, k) coinciding with the first and second fundamental forms of the embedding.

The bounded L^2 curvature conjecture, originally proposed by Klainerman in 1999 [17], roughly states that the initial value problem for the Einstein equations should be well-posed in the class of data (Σ, h, k) such that the Ricci curvature tensor of h and the first derivatives of k are in L^2_{loc} . Since the curvature tensor depends on the derivatives of the metric up to second order and since k encodes the data for its time derivative, this means that we should be able to control the solutions assuming only L^2 bounds on no more than two derivatives of the initial metric. This conjecture was recently proven by S. Klainerman, I. Rodnianski and J. Szeftel in the series of works

- S. KLAINERMAN, I. RODNIANSKI, J. SZEFTEL, *The bounded L^2 curvature conjecture*. arXiv:1204.1767, 91 p. (This is the main part of the series in which the proof is completed based on the results of the papers below.)
- J. SZEFTEL, *Parametrix for wave equations on a rough background I: regularity of the phase at initial time*. arXiv:1204.1768, 145 p.
- J. SZEFTEL, *Parametrix for wave equations on a rough background II: control of the parametrix at initial time*. arXiv:1204.1769, 84 p.
- J. SZEFTEL, *Parametrix for wave equations on a rough background III: space-time regularity of the phase*. arXiv:1204.1770, 276 p.
- J. SZEFTEL, *Parametrix for wave equations on a rough background IV: Control of the error term*. arXiv:1204.1771, 284 p.
- J. SZEFTEL, *Sharp Strichartz estimates for the wave equation on a rough background*. arXiv:1301.0112, 30 p.

The proof of the bounded L^2 curvature conjecture can be seen as the culminant point of a long sequence of works concerning the study of well-posedness for semi-linear and quasilinear wave equations applied to General Relativity and other geometric wave equations. This question has a long history, starting with the pioneer work of Choquet-Bruhat establishing existence and local uniqueness of solutions to (1) in the smooth case⁽⁴⁾. While we shall not try in this text to give an exhaustive treatment of the history of geometric wave equations with low regularity assumptions, the remaining of this introduction aims at providing enough information concerning its

⁽⁴⁾ In fact, the starting point of this history could arguably be placed around the 30's with the work of Darmois [12] on the Cauchy problem for the Einstein equations with analytic data and that of Stellmacher [39] on the uniqueness of solutions for smooth data. See also the work of Lichnerowicz [29].

developments so as to be able to highlight the main characteristics of the proof of the bounded L^2 curvature conjecture and in particular, explain why this result is the first of its kind. We start with some standard properties of the Einstein equations.

1.1. The Einstein equations as a system of quasilinear wave equations

The Einstein equations (1) being geometric, a choice of gauge, such as a choice of coordinates, is necessary so as to transform (1) into a system of partial differential equations amenable to various techniques from analysis. A popular choice for (1) is the *wave gauge* (also called harmonic or de Donder gauge). In the *wave gauge*, we consider a system of coordinates $(x^\mu)_{\mu=0,\dots,3}$ on a Lorentzian manifold (\mathcal{M}, g) such that for each μ , x^μ is a solution to the linear wave equation on (\mathcal{M}, g) , i.e., $\square_g x^\mu = 0$ where \square_g is the wave operator associated⁽⁵⁾ to g

$$\square_g := g^{\mu\nu} D_\mu D_\nu,$$

where D denotes the Levi-Civita connection associated to g . In wave gauge, the components of $\text{Ric}(g)$ simplify, so that (1) reduces to

$$(4) \quad \square_g g_{\mu\nu} = Q_{\mu\nu}(\partial g, \partial g),$$

where $Q_{\mu\nu}(\partial g, \partial g)$ denotes a quadratic form in the first derivatives of g with coefficients depending only on the components of g .

Since the principal symbol of \square_g , $g^{\mu\nu} \xi_\mu \xi_\nu$, is hyperbolic and depends on the solution itself, we see that the above system is a system of quasilinear wave equations.

1.2. The scale invariance of the equations

Note that the equations (4) are invariant under scaling: if g is a solution, then for any $\lambda > 0$, so is $(x^\mu) \rightarrow g(\lambda x^\mu)$. The 3-dimensional Sobolev space which is left invariant under this transformation is $\dot{H}^{s_c}(\mathbb{R}^3)$ with $s_c = 3/2$ being the critical exponent. Scaling symmetries play an important role in the study of non-linear wave equations. Roughly speaking, for H^s regularity with $s > s_c$, one can shrink the size of the data while making the time of existence larger, while for $s < s_c$, one typically expects ill-posedness.

⁽⁵⁾ In the whole text, the Einstein summation convention will be used. For instance, $g^{\mu\nu} D_\nu D_\mu$ stands for $\sum_{\mu,\nu=0,\dots,3} g^{\mu\nu} D_\mu D_\nu$.