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THE REGULARITY THEORY OF AREA-MINIMIZING INTEGRAL CURRENTS [after Almgren-De Lellis-Spadaro]

by **Luigi AMBROSIO**

INTRODUCTION

The development of Geometric Measure Theory has been mostly motivated by the attempts to solve Plateau's problem, that we can state here as follows.

Plateau's problem

Let M be an $(m+n)$-dimensional Riemannian manifold and let $\Gamma \subset M$ be a compact $(m-1)$-dimensional oriented embedded submanifold without boundary. Find an m-dimensional oriented embedded submanifold Σ with boundary Γ such that

$$\mathrm{vol}_m(\Sigma) \leq \mathrm{vol}_m(\Sigma'),$$

for all oriented submanifolds $\Sigma' \subset M$ such that $\partial\Sigma' = \Gamma$.

As a matter of fact, Plateau's problem (here stated in classical terms and for embedded submanifolds) can be very sensitive to the choice of the dimension m, the codimension n and to the class of admissible surfaces. For instance, in the case $m = 2$ and for boundaries Γ parametrized on the boundary of the unit disk D of $\mathbb{R}^2$, J. Douglas **[25]** and T. Radó **[44]** provided existence of solutions, using the fact that the so-called conformal parametrizations lead to good compactness properties of minimizing sequences. However, for general dimension and codimension, parametric methods fail.

It is a well-known fact that, in the formulation I gave, the solution of the Plateau problem does not always exist. For example, consider $M = \mathbb{R}^4$, $n = m = 2$ and Γ the smooth Jordan curve parametrized in the following way:

$$\Gamma = \big\{(\zeta^2, \zeta^3) : \zeta \in \mathbb{C},\ |\zeta| = 1\big\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

where we use the usual identification between $\mathbb{C}^2$ and $\mathbb{R}^4$, and we choose the orientation of Γ induced by the anti-clockwise orientation of the unit circle $|\zeta| = 1$ in $\mathbb{C}$.

It can be shown by a calibration technique (see the next sections) that there exists no smooth solution to the Plateau problem for such fixed boundary, and the *(singular) immersed* 2-dimensional disk

$$S = \{(z,w) \, : \, z^3 = w^2, \; |z| \le 1\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

oriented in such a way that $\partial S = \Gamma$, satisfies

$$\mathscr{H}^2(S) < \mathscr{H}^2(\Sigma),$$

for all smooth, oriented 2-dimensional submanifolds $\Sigma \subset \mathbb{R}^4$ with $\partial\Sigma = \Gamma$. Here and in the following we denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure, which for $k \in \mathbb{N}$ corresponds to the ordinary k-volume on smooth k-dimensional submanifolds.

This fact motivates the introduction of *weak solutions* to the Plateau problem, including at least immersed submanifolds, and the main questions about their existence and regularity. In this text I will focus on the line of thought that originated from E. De Giorgi's work for oriented hypersurfaces, thought as boundaries of sets (the so-called theory of sets of finite perimeter, closely related also to the theory of BV functions), which eventually led H. Federer and W.H. Fleming **[28]** to the very successful theory of currents, which provides weak solutions with no restriction on dimension and codimension (I will not discuss here the topological point of view and the formulation of Plateau's problem adopted by E.R. Reifenberg **[45, 46]**, which is more appropriate for non-oriented surfaces).

In a parallel way, also the regularity theory has been first developed in codimension 1, essentially thanks to the work of E. De Giorgi. The ideas introduced by E. De Giorgi in **[17]** had an impact also in other fields, as I will illustrate, and they could be almost immediately applied and adapted also to the higher codimension regularity theory (by F. Almgren **[7]** for currents, by B. Allard **[3]** for varifolds) to provide regularity *in a dense open subset* of the support. A major open problem, already pointed out in H. Federer's monograph **[26]**, was then the achievement of an *almost everywhere* regularity theory, possibly with an estimate on the codimension of the singular set $\mathrm{Sing}(T)$ inside the support $\mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$ out of the boundary.

It took many years to F. Almgren to develop an innovative and monumental program for the almost everywhere regularity and, at the same time, for the optimal estimate of the dimension of the singular set in arbitrary dimension and higher codimension. Announced in the early '80, circulated in preprint form and published posthumous in **[6]**, his work provides the interior partial regularity up to a (relatively) closed set of dimension at most $(m-2)$ inside $\mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$:

Theorem 0.1 (F. Almgren). — *Let T be an m-dimensional area minimizing integer rectifiable current in a C^5 Riemannian manifold M. Then, there exists a closed*