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DESIGNS EXIST!
[after Peter Keevash]

by **Gil KALAI**

INTRODUCTION

A set S of q-subsets of an n-set X is a *design* with parameters (n, q, r, λ) if every r-subset of X belongs to exactly λ elements of S. (The elements of S are called *blocks* and designs are also referred to as *block designs.*) There are some necessary *divisibility conditions* for the existence of such a design, namely that

$$\binom{q-i}{r-i} \text{ divides } \lambda\binom{n-i}{r-i}, \quad 0 \leq i \leq r-1. \tag{1}$$

To see that the divisibility conditions are necessary, fix any i-subset I of X and consider the sets in S that contain I.

The following result was conjectured in the 19th century and was recently proved by Peter Keevash.

Theorem 0.1 ([**29**]). — *For fixed $q, r,$ and λ, there exist $n_0(q, r, \lambda)$ such that if $n > n_0(q, r, \lambda)$ satisfies the divisibility conditions (1) then a design with parameters (n, q, r, λ) exists.*

In other words, for fixed q, r, and λ, the divisibility conditions are sufficient apart from a finite number of exceptional values of n.

A case of special interest is when $\lambda = 1$. A design of parameters $(n, q, r, 1)$ is called a *Steiner system* of parameters (n, q, r). The question if Steiner systems of given parameters exist goes back to works of Plücker, Kirkman, and Steiner. Until Keevash's result not a single Steiner system for $r > 5$ was known to exist.

The presentation of Keevash's work in this paper is based on Keevash's original paper [**29**], his Jerusalem videotaped lectures [**27**], and his subsequent paper [**28**]. It is also based on lecture notes and personal explanations by Jeff Kahn.

1. REGULARITY, SYMMETRY AND RANDOMNESS

1.1. Between regularity and symmetry

An object is "regular" if it looks locally the same (for a certain notion of "locality,"), and it is "symmetric" if it admits a transitive group action (on its "local" pieces). The interplay between regularity and symmetry is of interest in several parts of mathematics. For example, a regular graph is a graph where every vertex is adjacent to the same number of neighbors, and every graph whose group of automorphisms act transitively on its vertices is regular. Regular graphs need not be symmetric (most of them have no non-trivial automorphisms), but there are still various connections between general regular graphs and symmetry. Another example: manifolds are regular objects and Lie groups are symmetric types of manifolds; also here there are rich connections between regularity and symmetry.

Designs are regular objects. (The local pieces can be regarded as the r-subsets of the ground set.) You can get them from groups acting transitively on r-sets.

Proposition 1.1. — *Let Γ be a r-transitive permutation group. Then the orbit of a set of size q is a block design so that every set of size r belongs to the same number of blocks.*

However, it follows from the classification of finite simple groups that

Theorem 1.2. — *Let Γ be a r-transitive permutation group, $r > 5$, then G is A_n or S_n.*

1.2. The probabilistic method and quasi-randomness

The proof of the existence of designs is probabilistic. In order to prove the existence of objects of some kind satisfying a property **P**, one proves that for a suitable probability distribution on all objects of this kind there is a positive probability for property **P** to hold. The probabilistic method is of central importance in combinatorics (and other areas) [**1**]. Keevash defines a complicated combinatorial process with random ingredients for building a design, and shows that with positive probability it leads to the desired construction.

Quasi-randomness refers to deterministic properties of mathematical structures which allow them to behave (for certain restricted purposes) "as if they were random."(1) Quasi random properties of primes are, of course, of much importance. In graph theory, a sequence of graphs G_n (where G_n has n vertices) is quasi-random

(1) There are even cases that quasi-randomness of some kind can be attributed to arbitrary structures as is the case in Szemerédi regularity lemma which describes quasi-random structure on arbitrary graphs.